

WEIGHTED LAPLACE TRANSFORMS AND BESSEL FUNCTIONS ON HERMITIAN SYMMETRIC SPACES

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ABSTRACT. This paper defines π -weighted Laplace transforms on the spaces of π -covariant functions. By the inverse Laplace transform we define operator-valued Bessel functions. We also study the holomorphic discrete series of the automorphism group of a Siegel domain of type II.

1. INTRODUCTION

In this paper, we develop the theory of operator-valued Bessel functions on general Siegel domains, and apply these Bessel functions to study representations of their automorphism groups. The significance of this work can be described roughly as follows.

Bounded symmetric domains form a class of domains in several complex variables that generalize the ordinary unit disk in one variable. These domains are classified Lie theoretically as Hermitian symmetric spaces G/K having complex structure. For most of these domains the reductive Lie group G is a classical group, but in some cases the group G is “exceptional.” Moreover, certain of these domains are of “tube type”—that is, can be realized as half spaces in \mathbb{C}^n , while others are not. In a previous paper with K. Gross [DG1] we developed the theory of operator-valued Bessel functions for Hermitian symmetric spaces of tube type. We now complete that work by placing the theory in the general setting of an arbitrary Hermitian symmetric space, whether or not it is of tube type. Thus, the results in this paper provide a general construction, independent of domain by domain analysis, that includes not only the classical bounded symmetric domains but also the exceptional ones as well. In our work, it is convenient to frame our theory in terms of the unbounded realization of G/K as a generalized Siegel domain, realized in terms of the structure of Jordan pairs. For background on Jordan pairs, we refer to Section 2 below and [L1].

We now outline the main ideas in this paper.

Let (V, V^-) be a simple Jordan pair with a positive Hermitian involution. With respect to a maximal tripotent e , V is decomposed as $V = V_2 \oplus V_1$. Then V_2 is the complexification of a formally real Jordan algebra J , in which the interior

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Ω of sets of squares is the positive cone. With a positive definite bilinear map $B : V_1 \times V_1^- \rightarrow V_2$ we can define a Siegel domain Σ of type II by

$$(1) \quad \Sigma = \{x_2 + x_1 \in V_2 \oplus V_1 | \operatorname{Im} x_2 - \frac{1}{2}B(x_1, \bar{x}_1) > 0\}.$$

The Cayley transform c is a biholomorphic isomorphism of a circled bounded symmetric domain D with Σ .

Let G' and G be the groups of biholomorphic automorphisms of D and Σ , respectively, let K be the maximal compact subgroup of G' , and $K^\mathbb{C}$ the complexification of K . We decompose the Lie algebra \mathfrak{g} of G as a canonical graduation

$$(2) \quad \mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_{-\frac{1}{2}} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_{\frac{1}{2}} \oplus \mathfrak{g}_1,$$

and denote by G_{-1} , $G_{-\frac{1}{2}}$, G_0 , $G_{\frac{1}{2}}$, and G_1 the corresponding analytic subgroups. Let A be the connected component of the identity in G_0 , and let L be the maximal compact subgroup of A . We identify V_1 with a subgroup $K_{-\frac{1}{2}}$ of $K^\mathbb{C}$ by the map $u_0 \mapsto k(u_0)$. Note that as subgroups of $K^\mathbb{C}$, A normalizes $K_{-\frac{1}{2}}$ (cf. (4.8.2)), and $AK_{-\frac{1}{2}}$ is a subgroup of $K^\mathbb{C}$.

Let π be a finite-dimensional irreducible holomorphic representation of $K^\mathbb{C}$, denote by $\mathcal{V} = \mathcal{V}_\pi$ the space on which π acts and by $\mathcal{E} = \mathcal{E}_\pi$ the space of all linear transformations on \mathcal{V} . We will study \mathcal{V} -valued or \mathcal{E} -valued measurable functions f on $AK_{-\frac{1}{2}}$, which are holomorphic in $u_0 \in V_1$. f is called π -covariant if

$$(3) \quad f(ak(u_0)l) = \pi(l)^{-1}f(ak(u_0))$$

for all $a \in A$, $u_0 \in V_1$, and $l \in L$. Let Δ be the Koecher norm function and μ be a constant given in Section 3.4. Whenever the integral

$$(4) \quad \begin{aligned} F(z, u) &= \pi^{-\frac{n}{2}} \int_{AK_{-\frac{1}{2}}} e^{i(z|ae)} e^{(u|au_0)} e^{-(u_0|u_0)} \\ &\quad \times \pi(ak(iu_0))^{*-1} f(ak(u_0)) \Delta(ae)^\mu da du_0 \end{aligned}$$

is absolutely convergent for all $(z, u) \in \Sigma$, the function F on Σ is called the π -weighted Laplace transform of f .

The operator-valued Bessel function of weight π is the function $K_\pi : (AK_{-\frac{1}{2}}) \times V_1 \rightarrow \mathcal{E}$ obtained by Laplace inversion from the formula

$$(5) \quad \begin{aligned} &\int_{AK_{-\frac{1}{2}}} e^{i(z|ae)} e^{(u|au_0)} e^{-(u_0|u_0)} \pi(ak(iu_0))^{*-1} K_\pi(ak(u_0); v_0) da du_0 \\ &= e^{-i(z^{-1}|e)} e^{(z^{-1}|B(v_0, \bar{u}))} \pi(z) \pi(k(u)) \pi(k(-iv_0))^* \end{aligned}$$

for all $(z, u) \in \Sigma$ and $v_0 \in V_1$, where $\pi(z)$ is given by (4.2.1). Let $M = (m_1, m_2, \dots, m_r)$ with $m_1 \geq m_2 \geq \dots \geq m_r$ be the signature of the restricted lowest weight of π , and set $\omega(\pi) = m_r$. In this paper, we assume

$$(6) \quad \omega(\pi) > 2\mu + b - 1,$$

where b is given in Section 3.3. In this range, Laplace transform of (5) is valid, and the Bessel functions K_π also satisfy the square-integrability

$$(7) \quad \int_{(AK_{-\frac{1}{2}}) \times V_1} \|K_\pi(ak(u); v_0)\|^2 \Delta(ae)^{-\mu-b} e^{-(u|u)} e^{-(v_0|v_0)} da du dv_0 < \infty.$$

Let $\mathfrak{S}_\pi(AK_{-\frac{1}{2}})$ be the space of all π -covariant measurable functions $f : AK_{-\frac{1}{2}} \rightarrow \mathcal{V}_\pi$, which are holomorphic in u_0 and

$$(8) \quad \int_{AK_{-\frac{1}{2}}} \|f(ak(u_0))\|^2 e^{-(u_0|u_0)} da du_0 < \infty.$$

$\mathfrak{S}_\pi(AK_{-\frac{1}{2}})$ is a Hilbert space with respect to an inner product, and we denote the resulting Hilbert space by \mathfrak{S}_Γ . Using the Bessel function K_π , we define the Hankel transform \mathcal{K}_π on \mathfrak{S}_Γ by

$$(9) \quad (\mathcal{K}_\pi f)(ak(u_0)) = \int_{AK_{-\frac{1}{2}}} K_\pi(a_0^* ak(u_0); v_0) f(ak(v_0)) e^{-(v_0|v_0)} da_0 dv_0.$$

We construct a unitary operator from \mathfrak{S}_Γ to a Hilbert space \mathcal{H}_π of holomorphic \mathcal{V} -valued functions on Σ which intertwines two representations S_π and T_π of G indexed by finite-dimensional irreducible holomorphic representations π of $K^\mathbb{C}$. Denote by j the inversion element $(z, u) \mapsto (-z^{-1}, z^{-1} \circ u)$ of G . Then the operator $S_\pi(j)$ is the Hankel transform \mathcal{K}_π , and the square-integrability (7) of the Bessel functions K_π implies the square-integrability of S_π . We show also the irreducibility of the representations T_π (and of S_π). Then S_π (and also T_π) are in the holomorphic discrete series of G .

The paper is organized as follows.

In Section 2, we review the definitions, structures and properties of Jordan pairs and associated symmetric spaces. In Section 3, we study the automorphism group G of a Siegel domain Σ of type II, generating elements and the multipliers of G . Section 4 defines the operator-valued gamma function and the π -weighted Laplace transform. In Section 5, we define the operator-valued Bessel functions and study their square-integrability. In Section 6, we introduce the unitary equivalent Hilbert spaces \mathfrak{S}_Γ and \mathcal{H}_π , where \mathcal{H}_π are reproducing kernel spaces. We define the Hankel transforms by Bessel functions. In Section 7, we construct unitary representations S_π and T_π of G on \mathfrak{S}_Γ and \mathcal{H}_π . We prove the irreducibility and the square-integrability of S_π and T_π . We compute the reproducing kernel of \mathcal{H}_π explicitly.

We turn to the relationship of this paper to other literature.

In last decades, a number of mathematicians (cf. [B1], [Her], [Gi], [J], [GR], [Dib]) investigated Bessel functions on matrix spaces, and more generally, on symmetric cones. In [GK1] and [GK2], two kinds of operator-valued Bessel functions on matrix spaces were studied. In [DG1], we generalized the second or the so-called reduced Bessel functions from matrix spaces to symmetric cones in general by the Jordan algebra structure, and applied these functions to study representations of automorphism groups of all Hermitian symmetric spaces of tube type. In [Di1]–[Di3], we developed a theory of operator-valued Bessel functions on mixed Schrödinger-Fock spaces, and studied the harmonic representations and holomorphic discrete series for the groups $U(p, q)$ and $O_*(2m)$ with m odd. These groups are known as automorphism groups of certain Hermitian symmetric spaces, which are not of tube type and are called Siegel domains of type II.

Upon these stepping stones, we now broaden the scope of Bessel function theory to all Siegel domains by the structure of Jordan pairs and the theory of Siegel domains, which were developed by a number of authors (cf. [L1]–[L3], [Do1], [Do2], [Gi], [Kan], [M], [P]). We study also representations of the automorphism group of Σ .

As related works, the harmonic representation of certain groups was studied in [Sh], [Se] and [We]. [KV] studied the harmonic representation of $U(p, q)$ by pluriharmonic polynomials. [Ho1], [Ho2] developed the theory of dual reductive pairs.

In the next paper [DG2], we will consider the analytic continuation of the Bessel function and analytic continuation of the holomorphic discrete series for the universal covering group of G , as well as other spaces in the Wallach set (cf. [W1], [W2], [EHW]) associated to representations π . We will also study the operator-valued hypergeometric functions on Hermitian symmetric spaces.

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2. JORDAN PAIRS AND SYMMETRIC SPACES

In this section, we review briefly the definitions and structures of Jordan pairs and associated symmetric spaces, and list the properties that are needed in this paper. We refer to [L1] and [FK2] for more detail.

2.1. Jordan pairs. Let V^+ and V^- be complex vector spaces, and let $Q_+ : V^+ \rightarrow \text{Hom}(V^-, V^+)$ and $Q_- : V^- \rightarrow \text{Hom}(V^+, V^-)$ be quadratic maps; i.e., $Q_+(\alpha v) = \alpha^2 Q_+(v)$ and $Q_-(\alpha v^-) = \alpha^2 Q_-(v^-)$ for $\alpha \in \mathbb{C}$, $v \in V^+$ and $v^- \in V^-$. Define trilinear maps $\{ \cdot \cdot \cdot \} : V^\sigma \times V^{-\sigma} \times V^\sigma \rightarrow V^\sigma$ and bilinear maps $D_\sigma : V^\sigma \times V^{-\sigma} \rightarrow \text{End}(V^\sigma)$ by

$$(1) \quad \{xyz\} = D_\sigma(x, y)z = Q_\sigma(x + z)y - Q_\sigma(x)y - Q_\sigma(z)y$$

for $x, z \in V^\sigma$, $y \in V^{-\sigma}$, $(\sigma = \pm)$. The pair (V^+, V^-) together with the quadratic maps (Q_+, Q_-) is called a *Jordan pair* if the identity

$$(2) \quad \{uv\{xyz\}\} - \{xy\{uvz\}\} = \{\{uvx\}yz\} - \{x\{vuy\}z\}$$

holds for all $u, x, z \in V^\sigma$, $v, y \in V^{-\sigma}$, $\sigma = \pm$.

A *Hermitian involution* of a complex Jordan pair is an invertible complex anti-linear map $\delta : V^+ \rightarrow V^-$ such that $\delta(Q_+(x)\delta y) = Q_-(\delta x)y$ for all $x, y \in V^+$. We say δ is *positive* if $Q_+(x)\delta x = \lambda x$ for $\lambda \in \mathbb{C}$ implies $\lambda > 0$ for all non-zero $x \in V^+$.

2.2. Jordan algebras. A (linear) *Jordan algebra* is a vector space J having a unit e with a commutative bilinear multiplication $x \circ y$ satisfying $x^2 \circ (x \circ y) = x \circ (x^2 \circ y)$, where $x^2 = x \circ x$. The *quadratic representation* P of J is defined by

$$(1) \quad P(x)y = \frac{1}{2}x \circ (x \circ y) - \frac{1}{4}x^2 \circ y$$

for all $x, y \in J$. (Note: Here, the product $x \circ y$ is twice of what it is in some literature. By this assumption, (2.7.5) below is simpler.) The representation P of J can be extended to its complexification $J^\mathbb{C}$. $z \in J^\mathbb{C}$ is invertible if and only if $P(z)$ is invertible and then

$$(2) \quad P(z^{-1}) = P(z)^{-1}.$$

By [Koe], VII.1(7),

$$(3) \quad P(z^*)[P(\text{Im } z)]^{-1}P(z) = [P(\text{Im } (-z^{-1}))]^{-1}$$

and

$$(4) \quad P(z)[P(\text{Im } z)]^{-1}P(z^*) = [P(\text{Im } (-z^{-1}))]^{-1}$$

for invertible $z \in J^{\mathbb{C}}$, where z^* is the complex conjugate of z in $J^{\mathbb{C}}$ and $\operatorname{Im} z = \frac{1}{2i}(z - z^*)$. A real Jordan algebra is *formally real* if $x^2 + y^2 = 0$ implies $x = y = 0$. Every Jordan algebra J gives rise to a Jordan pair by setting $V^+ = V^- = J$ and $Q_+(x)y = Q_-(x)y = P(x)y$ for $x, y \in J$.

A non-empty, open, self-dual, homogeneous cone in a finite-dimensional Euclidean space is called a *symmetric cone*. In a formally real Jordan algebra J the interior Ω of the set $\{x^2 | x \in J\}$ of all squares is a symmetric cone, and any symmetric cone is isomorphic to a cone arising from a formally real Jordan algebra in this way. The elements of Ω are invertible in J , and Ω is the connected component of e in the set of invertible elements of J .

2.3. Tripotents. To simplify the notation, we omit the subscript and superscript $+$; i.e., we write V, Q, D for V^+, Q_+, D_+ , respectively. An *idempotent* in a Jordan pair (V, V^-) is a pair $(a, b) \in V \times V^-$ such that $Q(a)b = a$ and $Q_-(b)a = b$. If (V, V^-) has a Hermitian involution δ and denotes $\delta v = \bar{v}$, then the set of all real points of the idempotent set is the set of all idempotents of the form (e, \bar{e}) and thus may be identified (by projection onto the first factor) with the set M of e in V satisfying $Q(e)\bar{e} = e$. Such an element $e \in V$ is called a *tripotent*; i.e., e is called a tripotent if $Q(e)\bar{e} = e$.

2.4. Peirce decomposition. Let (V, V^-) be a Jordan pair with the Hermitian involution δ , $\delta x = \bar{x}$, and let e be a tripotent of V . Then we have a *Peirce decomposition* of V with respect to e .

(a) V decomposes as

$$(1) \quad V = V_2 \oplus V_1 \oplus V_0$$

where $V_\alpha = V_\alpha(e)$ is the α -eigenspace of $D(e, \bar{e})$. The V_α are orthogonal with respect to any associative inner product and satisfy the multiplication rules

$$(2) \quad \{V_\alpha \bar{V}_\beta V_\gamma\} \subset V_{\alpha-\beta+\gamma}, \quad \{V_2 \bar{V}_0 V\} = \{V_0 \bar{V}_2 V\} = 0.$$

In particular, $(V_\alpha, \bar{V}_\alpha)$ is a δ -invariant subpair for $\alpha = 0, 1, 2$.

(b) V_2 is a complex Jordan algebra with Jordan product $x \circ y = \{x\bar{e}y\}$ and the *unit element* e . The map $z \mapsto z^* = Q(e)\bar{z}$ is a complex antilinear automorphism of period 2 of the Jordan algebra V_2 .

(c) The fixed point set $J = J(e)$ of the map $z \mapsto z^*$ is a real Jordan algebra, and $V_2 = J \oplus iJ$ is the complexification of J . If δ is positive, then J is formally real.

2.5. Circled bounded symmetric domains. Let D be a bounded domain in a finite-dimensional complex vector space V . D is called *symmetric* if, for every $z \in D$, there exists an automorphism S_z of period two of D , having z as an isolated fixed point. D is called *circled* (with respect to 0) if $0 \in D$, and $e^{it}z \in D$ for all $z \in D, t \in \mathbb{R}$.

Every bounded symmetric domain in V is isomorphic to a bounded symmetric and circled domain which is unique up to a linear isomorphism of V .

Let (V, V^-) be a Jordan pair with a positive Hermitian involution. Then every $v \in V$ can be written uniquely as

$$(1) \quad v = \lambda_1 e_1 + \lambda_2 e_2 + \cdots + \lambda_n e_n.$$

Here the e_i are pairwise orthogonal non-zero tripotents which are real combinations of powers of v , and the λ_i satisfy

$$(2) \quad 0 \leq \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n.$$

We call (1) the spectral decomposition of v and the λ_i the eigenvalues of v .

Let D be a circled bounded symmetric domain. Then D is the open unit ball of the spectral norm of the associated Jordan pair (V, V^-) with the positive Hermitian involution $\delta : v \mapsto \bar{v}$, where the spectral norm $|v|$ of $v \in V$ denotes the largest eigenvalue of v . Conversely, given a Jordan pair with the positive Hermitian involution, the open unit ball of the spectral norm is a circled bounded symmetric domain whose associated Jordan pair is the given one.

2.6. Simple Jordan pairs. A Jordan pair (V, V^-) is called *simple* if the Q -operators are non-trivial, and if it contains no proper ideals. Here an ideal of (V, V^-) is a pair (I, I^-) of subspaces such that $\{IV^-V\} + \{VI^-V\} \subset I$ and $\{I^-VV^-\} + \{V^-I^-V^-\} \subset I^-$. A finite-dimensional semisimple Jordan pair is the direct sum of simple ideals which are unique up to order.

The corresponding concept for domains is irreducibility. A circled bounded symmetric domain D is called *irreducible* if it is not isomorphic to a direct product $D' \times D''$ of lower-dimensional circled bounded symmetric domains. The domain D is irreducible if and only if (V, V^-) is simple. The classification of simple Jordan pairs with positive involutions can be found in [L1].

2.7. Siegel domains of type II. As in Section 2.5, let D be a circled bounded symmetric domain, G' its automorphism group, K the isotropy subgroup of 0 in G' , and let $(\cdot|\cdot)$ be a K -invariant Hermitian inner product on V . We denote by M the set of all tripotents of V , and define an ordering on M by

$$c < e \iff e - c \in M \quad \text{and} \quad c \perp (e - c).$$

A tripotent e is *maximal* with respect to this ordering if and only if the Peirce space $V_0(e) = 0$.

Let (V, \bar{V}) be a Jordan pair with the Hermitian involution $v \mapsto \bar{v}$, and e be a maximal tripotent of V . Since $V_0(e) = 0$, $x = x_2 + x_1$ is the decomposition of $x \in D$ in the Peirce spaces, where $x_2 \in V_2(e)$ and $x_1 \in V_1(e)$. Let

$$(1) \quad \Sigma = \{x_2 + x_1 \in V_2 \oplus V_1 \mid \operatorname{Im} x_2 - \frac{1}{2}B(x_1, \bar{x}_1) > 0\},$$

where $\operatorname{Im} x_2 = \frac{1}{2i}(x_2 - x_2^*)$ and the bilinear map $B : V_1 \times \bar{V}_1 \rightarrow V_2$ is defined by

$$B(u, \bar{v}) = \{u\bar{v}e\} \quad (u \in V_1, \bar{v} \in \bar{V}_1).$$

Σ is called a *Siegel domain of type II*. Note that B is Hermitian and positive definite in the sense that

$$B(u, \bar{v})^* = B(v, \bar{u}),$$

$$B(u, \bar{u}) \geq 0,$$

and $B(u, \bar{u}) = 0$ only if $u = 0$. For $z \in V_2$, $u \in V_1$, define

$$(2) \quad z \circ u = \{z\bar{e}u\}.$$

Let $x = x_2 + x_1 \in V_2 \oplus V_1$. The Cayley transform c given by

$$(3) \quad c(x) = \frac{1}{2}(x_2 + ie) \circ (ix_2 + e)^{-1} + \sqrt{2}(ix_2 + e)^{-1} \circ x_1$$

is a biholomorphic isomorphism of D with Σ .

By [L1], Proposition 10.11, we have

$$(4) \quad P(z)B(u, \bar{v}) = B(z \circ u, \overline{z^* \circ v}),$$

and

$$(5) \quad z \circ B(u, \bar{v}) = B(z \circ u, \bar{v}) + B(u, \overline{z^* \circ v})$$

for all $z \in V_2$, $u, v \in V_1$.

By [Do1], (2.7),

$$(6) \quad (P(z)w) \circ u = z \circ (w \circ (z \circ u))$$

for all $z, w \in V_2$, $u \in V_1$. By (2.2.3), (2.2.4) and (4), we have

$$(7) \quad z \circ ((\operatorname{Im} z)^{-1} \circ (z^* \circ u)) = \operatorname{Im}(-z^{-1}) \circ u$$

and

$$(8) \quad z^* \circ ((\operatorname{Im} z)^{-1} \circ (z \circ u)) = \operatorname{Im}(-z^{-1}) \circ u$$

for all $u \in V_1$ and invertible $z \in V_2$.

2.8. Remark. In this paper, we study the circled bounded symmetric domain D and the Siegel domain Σ of type II associated with a simple Jordan pair (V, V^-) . (V, V^-) has a positive Hermitian involution $v \mapsto \bar{v}$. e is a maximal tripotent of V , $V_0(e) = 0$ and $V = V_2 \oplus V_1$. Then V_2 is the complexification of a formally real Jordan algebra J , in which the interior Ω of sets of squares is the positive cone, e is the unit element of J , and $(\cdot|\cdot)$ is a K -invariant inner product on V (in particular, on V_2). We assume that Ω is an irreducible symmetric cone in the simple Jordan algebra J .

3. AUTOMORPHISM GROUPS AND LIE ALGEBRA STRUCTURES

In this section, we recall the properties of automorphism groups of circled bounded symmetric domains D and Siegel domains Σ of type II, their Lie algebras and complexifications, which are important for the sequel. We refer to [FK1], [RV1], and [Do1] for more detail. We also study generating elements and multipliers for the automorphism group of Σ .

3.1. Automorphism group of D . Let G' be the automorphism group of D and \mathfrak{g}' its Lie algebra. Let us consider the Cartan decomposition $\mathfrak{g}' = \mathfrak{k} + \mathfrak{p}$, where \mathfrak{k} has non-trivial center and \mathfrak{h} will be a Cartan subalgebra of \mathfrak{k} (and hence also of \mathfrak{g}'). $\mathfrak{g}^{\mathbb{C}}$, $\mathfrak{k}^{\mathbb{C}}$, $\mathfrak{p}^{\mathbb{C}}$, $\mathfrak{h}^{\mathbb{C}}$ are their complexifications, respectively.

The $\mathfrak{h}^{\mathbb{C}}$ -roots of $\mathfrak{g}^{\mathbb{C}}$ that are also roots of $\mathfrak{k}^{\mathbb{C}}$ are called compact roots. We denote by Φ^+ the set of positive non-compact roots. Denoting by τ the conjugation with respect to the real form $\mathfrak{k} + i\mathfrak{p}$, we consider a basis of root vectors e_{α} such that $\tau e_{\alpha} = -e_{-\alpha}$, $[e_{\alpha}, e_{-\alpha}] = h_{\alpha}$, $[h_{\alpha}, e_{\pm\alpha}] = \pm 2e_{\pm\alpha}$. Setting

$$\mathfrak{p}^{\pm} = \sum_{\alpha \in \Phi^+} \mathbb{C} e_{\pm\alpha},$$

we have

$$(1) \quad \mathfrak{g}^{\mathbb{C}} = \mathfrak{p}^- + \mathfrak{k}^{\mathbb{C}} + \mathfrak{p}^+.$$

On \mathfrak{p}^+ we have a Hermitian inner product defined by

$$(2) \quad (z|w) = -K(z, \tau w)$$

where K is the Killing form.

$\gamma_1, \gamma_2, \dots, \gamma_r$ will be strongly orthogonal roots of Harish-Chandra, with the ordering $\gamma_1 > \gamma_2 > \dots > \gamma_r$. r is the real rank of \mathfrak{g}' . We will use the abbreviated notations

$$e_j = e_{\gamma_j}, \quad h_j = h_{\gamma_j} \quad (1 \leq j \leq r), \quad e = \sum_1^r e_j.$$

We denote by \mathfrak{h}^- the real span of vectors ih_j , by \mathfrak{a}^+ the real span of the e_j , and by \mathfrak{a} the real span of the vectors $i(e_j + \tau e_j)$, $(1 \leq j \leq r)$. \mathfrak{a} is a Cartan subalgebra of the pair $(\mathfrak{g}', \mathfrak{k})$.

$G^{\mathbb{C}}$ will be the adjoint group of $\mathfrak{g}^{\mathbb{C}}$. G' , K , $K^{\mathbb{C}}$, P^{\pm} , A will be analytic subgroups corresponding to \mathfrak{g}' , \mathfrak{k} , $\mathfrak{k}^{\mathbb{C}}$, \mathfrak{p}^{\pm} , \mathfrak{a} . We identify \mathfrak{p}^+ with P^+ by the exponential map.

3.2. Cayley transform. Let c be the Cayley transform given by (2.7.3). We write G for $cG'c^{-1}$ and \mathfrak{g} for its Lie algebra. Let \mathfrak{g}'_T be the fixed point set of \mathfrak{g}' under $Ad(c^4)$ and $\mathfrak{g}_T^{\mathbb{C}}$ its complexification. The intersections of \mathfrak{k} , \mathfrak{p} , \mathfrak{p}^+ with $\mathfrak{g}_T^{\mathbb{C}}$ give corresponding decompositions $\mathfrak{g}'_T = \mathfrak{k}_T + \mathfrak{p}_1$ and $\mathfrak{g}_T^{\mathbb{C}} = \mathfrak{p}_1^- + \mathfrak{k}_T^{\mathbb{C}} + \mathfrak{p}_1^+$.

$\mathfrak{n}_1^{\pm} = \mathfrak{g} \cap \mathfrak{p}_1^{\pm}$ is a real form of \mathfrak{p}_1^{\pm} . $Ad(c^2)$ is a Cartan involution of \mathfrak{k}_T ; the corresponding decomposition is $\mathfrak{k}_T = \mathfrak{l}_T + \mathfrak{q}_1$. Writing $\mathfrak{k}_T^* = \mathfrak{l}_T + i\mathfrak{q}_1$, K_T^* and L_T for the corresponding analytic groups of \mathfrak{k}_T^* and \mathfrak{l}_T , and L_T^0 for the connected component of the identity in L_T , the orbit $\Omega = K_T^* \cdot e \cong K_T^*/L_T^0$ is a homogeneous selfdual cone in \mathfrak{n}_1^+ .

\mathfrak{n}_1^+ has the structure of a formally real Jordan algebra J , in which Ω is the interior of the set of squares, e is a maximal tripotent of V , \mathfrak{p}_1^+ coincides with V_2 , \mathfrak{p}^+ coincides with $V = V_2 \oplus V_1$ (cf. Section 2.4), and the inner product defined by (3.1.2) coincides with one given in Section 2.7.

The Cayley transform \mathfrak{g}_T of \mathfrak{g}'_T has the vector space decomposition

$$(1) \quad \mathfrak{g}_T = \mathfrak{n}_1^- + \mathfrak{k}_T^* + \mathfrak{n}_1^+.$$

We define \mathfrak{m} to be the centralizer of $i\mathfrak{h}^-$ in \mathfrak{l}_T ; it is then also the centralizer of $i\mathfrak{h}^-$ in \mathfrak{g}_T and \mathfrak{g} .

3.3. Restricted root space decompositions. We have $i\mathfrak{h}^- \subset i\mathfrak{q}_1 \subset \mathfrak{g}_T$, and, since $Ad(c)$ interchanges $i\mathfrak{h}^-$ with \mathfrak{a} , $i\mathfrak{h}^-$ is a Cartan subalgebra of the pair $(\mathfrak{g}, \mathfrak{k})$ as well as of $(\mathfrak{k}_T^*, \mathfrak{l}_T)$. The $i\mathfrak{h}^-$ -roots of \mathfrak{g} are $\pm \frac{1}{2}(\gamma_j \pm \gamma_k)$, $\pm \gamma_j$, $\pm \frac{1}{2}\gamma_j$, $(1 \leq j, k \leq r)$ with respective multiplicities ν , 1 and $2b$ (independent of j, k). The corresponding (restricted) root spaces refine the decompositions (3.2.1) and (3.1.1) as follows:

$$(1) \quad \mathfrak{n}_1^+ = \sum_{j < k} \mathfrak{n}^{+jk} + \sum_j \mathfrak{n}^{+j},$$

$$(2) \quad \mathfrak{p}_{1/2}^+ = \sum_j \mathfrak{p}^{+j/2},$$

$$(3) \quad \mathfrak{p}^+ = \mathfrak{p}_1^+ + \mathfrak{p}_{1/2}^+,$$

where \mathfrak{n}^{+jk} , \mathfrak{n}^{+j} are root spaces in \mathfrak{g} for $\frac{1}{2}(\gamma_j + \gamma_k)$ and γ_j , respectively, and $\mathfrak{p}^{+j/2}$ is the root space in $\mathfrak{p}^+(\subset \mathfrak{g}^{\mathbb{C}})$ for $\frac{1}{2}\gamma_j$. Furthermore,

$$(4) \quad \mathfrak{n}_K = \sum_{j < k} \mathfrak{n}_K^{jk}, \quad \bar{\mathfrak{n}}_K = \sum_{j > k} \mathfrak{n}_K^{jk},$$

$$(5) \quad \mathfrak{k}_T^* = i\mathfrak{h}^- + \mathfrak{m} + \mathfrak{n}_K + \bar{\mathfrak{n}}_K,$$

$$(6) \quad \mathfrak{k}^{\mathbb{C}} = (\mathfrak{k}_T^*)^{\mathbb{C}} + \sum_j \mathfrak{k}^{j/2} + \sum_j \mathfrak{k}^{-j/2},$$

where \mathfrak{n}_K^{jk} is the root space for $\frac{1}{2}(\gamma_j - \gamma_k)$ in \mathfrak{g} and $\mathfrak{k}^{\pm j/2}$ is the root space for $\pm \frac{1}{2}\gamma_j$ in $\mathfrak{k}^{\mathbb{C}}$. (The spaces $\mathfrak{p}^{\pm j/2}$, $\mathfrak{k}^{\pm j/2}$ do not meet \mathfrak{g} ; the $\frac{1}{2}\gamma_j$ -root space in \mathfrak{g} is a real form of $\mathfrak{p}^{+j/2} + \mathfrak{k}^{+j/2}$.)

3.4. Koecher norm function. Let $n = \dim_{\mathbb{C}} \mathfrak{p}^+$, $n_1 = \dim_{\mathbb{C}} \mathfrak{p}_1^+$. Now r is the rank of Ω , and n_1 is the (real) dimension of J . A dimension count gives

$$(1) \quad n_1 = \frac{r(r-1)}{2}\nu + r,$$

$$(2) \quad n = n_1 + rb.$$

Set

$$(3) \quad \mu = n_1/r.$$

The number μ is an integer or half-integer, and

$$(4) \quad \mu = \frac{r-1}{2}\nu + 1.$$

With respect to the inner product $(\cdot|\cdot)$, the *characteristic function* ϕ of Ω is defined by

$$(5) \quad \phi(x) = \int_{\Omega} e^{-(x|y)} dy$$

for $x \in \Omega$, from which the *Koecher norm function* Δ on Ω is given by

$$(6) \quad \Delta(x) = c\phi(x)^{-1/\mu}$$

where c is a constant chosen so that $\Delta(e) = 1$.

3.5. The polar decomposition of J . Let

$$(1) \quad \mathfrak{R} = \left\{ s = \sum_{j=1}^r s_j e_j : s_j \in \mathbb{R} \right\}$$

and

$$(2) \quad \mathfrak{R}^+ = \{s \in \mathfrak{R} : s_1 > \cdots > s_r\}.$$

Any element $x \in J = \mathfrak{n}_1^+$ can be written as

$$(3) \quad x = l \cdot s$$

where $s \in \bar{\mathfrak{R}}^+$ is unique and $l \in L_T^0$ is unique modulo the centralizer of \mathfrak{R} in L_T^0 . Then

$$(4) \quad \Delta\left(l \cdot \sum_{j=1}^r s_j e_j\right) = \prod_{j=1}^r s_j.$$

The mapping $L_T^0 \times \bar{\mathfrak{R}}^+ \rightarrow J$ given by (3) defines the *polar decomposition* for J . The corresponding integral formula is

$$(5) \quad \int_J f(x) dx = c \int_{\mathfrak{R}^+} \left(\int_{L_T^0} f(l \cdot s) dl \right) \prod_{j < k} (s_j - s_k)^\nu ds_1 \cdots ds_r$$

where dx is the Euclidean volume element on J , dl is the normalized Haar measure on L_T^0 , c is a positive constant depending only upon J , and f is any integrable function.

Similarly, let dx again be the Euclidean volume element on Ω and dl the normalized Haar measure on L_T^0 , then

$$(6) \quad \int_\Omega f(x) dx = c_0 \int_0^\infty \cdots \int_0^\infty \int_{L_T^0} f(l \cdot \sum_1^r t_j e_j) dl \prod_{j < k} |t_j - t_k|^\nu dt_1 \cdots dt_r,$$

where c_0 is also a constant.

Let dz be the Euclidean volume element on \mathfrak{p}^+ and dk the normalized Haar measure on K , then

$$(7) \quad \begin{aligned} \int_{\mathfrak{p}^+} f(z) dz &= c \int_0^\infty \cdots \int_0^\infty \int_K f(k \cdot \sum_1^r t_j e_j) dk 2^r \\ &\quad \times \prod_j t_j^{2b+1} \prod_{j < k} |t_j^2 - t_k^2|^\nu dt_1 \cdots dt_r, \end{aligned}$$

where c is a constant.

3.6. Generating elements of G . As in [Do1], we decompose \mathfrak{g} as a canonical graduation

$$(1) \quad \mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_{-\frac{1}{2}} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_{\frac{1}{2}} \oplus \mathfrak{g}_1.$$

Then \mathfrak{k}_T^* in Section 3.2 coincides with \mathfrak{g}_0 (cf. [M]), \mathfrak{n}_1^\pm coincides with $\mathfrak{g}_{\pm 1}$,

$$(2) \quad \mathfrak{g}_{-\frac{1}{2}} = \sum_{i=1}^r \mathfrak{g}_{-\frac{1}{2}\gamma_i},$$

and

$$(3) \quad \mathfrak{g}_{\frac{1}{2}} = \sum_{i=1}^r \mathfrak{g}_{\frac{1}{2}\gamma_i},$$

where $\mathfrak{g}_{\pm \frac{1}{2}\gamma_i}$ are the root spaces in \mathfrak{g} for $\pm \frac{1}{2}\gamma_i$.

We adopt a notation (z, u) for the element of $V_2 \oplus V_1$, where $z \in V_2$ and $u \in V_1$; i.e., $(z, u) = z + u$. Denote G_{-1} , $G_{-\frac{1}{2}}$, G_0 , $G_{\frac{1}{2}}$, and G_1 as the corresponding analytic subgroups. Viewing G as an automorphism group of the domain Σ given by (2.7.1), G_0 , $G_{\frac{1}{2}}$, and G_1 act on Σ as follows:

(a) Elements a in G_0 can be described as

$$(4) \quad a \cdot (z, u) = (az, au);$$

i.e., as an operator on V_2 , $a \in GL(V_2)$ and $a\Omega = \Omega$; as an operator on V_1 , $a \in GL(V_1)$; moreover,

$$(5) \quad aB(u, \bar{u}) = B(au, \overline{au}).$$

(b) Elements in $G_{\frac{1}{2}}$ can be parametrized by elements u_0 in V_1 ; i.e., for $u_0 \in V_1$

$$(6) \quad R(u_0) \cdot (z, u) = (z + iB(u, \bar{u}_0) + \frac{i}{2}B(u_0, \bar{u}_0), u + u_0).$$

(c) Elements in G_1 can be parametrized by elements x_0 in J ; i.e., for $x_0 \in J$

$$(7) \quad R(x_0) \cdot (z, u) = (z + x_0, u).$$

By [Do1, Corollary 3.6] the symmetry g_{ie} of the Siegel domain Σ with respect to the point ie is given by

$$g_{ie} \cdot (z, u) = (-z^{-1}, -iz^{-1} \circ u),$$

where $z \circ u$ is given by (2.7.2). To simplify the discussion, we study the element j given by

$$(8) \quad j \cdot (z, u) = (-z^{-1}, -z^{-1} \circ u).$$

Denote by θ the Cartan involution with respect to the point ie , then $\theta = Ad(g_{ie})$. By [Do1], $\theta \mathfrak{g}_\lambda = \mathfrak{g}_{-\lambda}$ for $\lambda \in \{-1, -\frac{1}{2}, 0, \frac{1}{2}, 1\}$. By (1), elements a , $R(u_0)$, $R(x_0)$ and g_{ie} generate the group G . Then by (8), a , $R(u_0)$, $R(x_0)$ and j generate the group G .

3.7. The multiplier m . For $g \in G$ denote by $\frac{\partial g \cdot (z, u)}{\partial(z, u)}$ the Jacobi at $(z, u) \in \Sigma$; i.e.,

$$(1) \quad \frac{\partial g \cdot (z, u)}{\partial(z, u)}(w, v) = \lim_{h \rightarrow 0} \frac{g \cdot (z + hw, u + hv) - g \cdot (z, u)}{h}$$

for $(w, v) \in V_2 \oplus V_1$, where h is a complex scalar. Define

$$(2) \quad m(g, (z, u)) = \left(\frac{\partial g^{-1} \cdot (z, u)}{\partial(z, u)} \right)^{-1}$$

for $(g, (z, u)) \in G \times \Sigma$. Then m satisfies the *multiplier identity*

$$(3) \quad m(g_1 g_2, (z, u)) = m(g_1, (z, u)) m(g_2, g_1^{-1} \cdot (z, u))$$

for $g_1, g_2 \in G$ and $(z, u) \in \Sigma$. The definitions (1) and (2) can be generalized to $g \in G^{\mathbb{C}}$, $(z, u) \in V_2 \oplus V_1$, and the property (3) is still valid.

We compute m on the set of generators a , $R(u_0)$, $R(x_0)$, j of G given by (3.6.4), (3.6.6), (3.6.7) and (3.6.8).

3.8. Proposition. For the generators of G given by (3.6.4), (3.6.6), (3.6.7) and (3.6.8),

$$(1) \quad m(a, (z, u)) = a,$$

$$(2) \quad m(R(u_0), (z, u)) \cdot (w, v) = (w + iB(v, \bar{u}_0), v),$$

$$(3) \quad m(R(x_0), (z, u)) = I,$$

$$(4) \quad m(j, (z, u)) \cdot (w, v) = (P(z)w, z \circ v + z \circ (w \circ u)).$$

Proof. (1) and (3) follow from direct calculations. Since

$$\begin{aligned} & \frac{\partial(R(u_0) \cdot (z, u))}{\partial(z, u)}(w, v) \\ &= \lim_{h \rightarrow 0} \left(\frac{z + hw + iB(u + hv, \bar{u}_0) + \frac{i}{2}B(u_0, \bar{u}_0) - z - iB(u, \bar{u}_0) - \frac{i}{2}B(u_0, \bar{u}_0)}{h}, \right. \\ & \quad \left. \frac{u + hv + u_0 - u - u_0}{h} \right) \\ &= (w + iB(v, \bar{u}_0), v), \end{aligned}$$

we have (2). By [Koe, p. 64],

$$\begin{aligned} & \frac{\partial(j^{-1} \cdot (z, u))}{\partial(z, u)}(w, v) \\ &= \lim_{h \rightarrow 0} \left(\frac{-(z + hw)^{-1} + z^{-1}}{h}, \frac{(z + hw)^{-1} \circ (u + hv) - z^{-1} \circ u}{h} \right) \\ &= (P(z)^{-1}w, z^{-1} \circ v - (P(z)^{-1}w) \circ u). \end{aligned}$$

Then (4) follows from (3.7.2).

For $z_0 \in V_2$, define a mapping $r(z_0)$ by

$$(5) \quad r(z_0) : (z, u) \mapsto (P(z_0)z, z_0 \circ u).$$

3.9. Proposition. *Operators a , $m(R(u_0), (z, u))$, $m(R(x_0), (z, u))$, $m(j, (z, u))$ given by (3.8.1)–(3.8.4) are elements in $K^{\mathbb{C}}$.*

Proof. By Section 3.6, $a \in K_T^* \subset K_T^{\mathbb{C}} \subset K^{\mathbb{C}}$. Denote

$$\mathfrak{k}_{\frac{1}{2}} = \sum_{j=1}^r \mathfrak{k}^{j/2}, \quad \mathfrak{k}_{-\frac{1}{2}} = \sum_{j=1}^r \mathfrak{k}^{-j/2},$$

and $K_{\frac{1}{2}}$, $K_{-\frac{1}{2}}$ as the corresponding subgroups. Then $K_{-\frac{1}{2}}$ can be parametrized by elements $u_0 \in V_1$; i.e.,

$$(1) \quad k(u_0) : (z, u) \mapsto (z, z \circ u_0 + u)$$

where $u_0 \in V_1$. By [L1, 3.4, 3.5], $k(u_0)^*$ is the map

$$(2) \quad k(u_0)^* : (z, u) \mapsto (z + B(u, \bar{u}_0), u),$$

and $k(u_0)^* \in K_{\frac{1}{2}}$. Thus, (1) and (2) are in $K^{\mathbb{C}}$, so is $m(R(u_0), (z, u))$. It is obvious that $m(R(x_0), (z, u)) \in K^{\mathbb{C}}$.

By [L1], 10.11(2), $r(x_0)$ given by (3.8.5) is in K_T^* for invertible $x \in J$. Consequently, $r(z_0) \in K^{\mathbb{C}}$ for all invertible $z_0 \in V_2$. If $(z, u) \in \Sigma$, then $\text{Im } z > 0$ and z is invertible. By (1) and (3.8.4), $m(j, (z, u)) \in K^{\mathbb{C}}$ and the proposition is proved.

3.10. Decompositions of A . We now study the connected components of the identity in K_T^* and L_T , and denote them by A and L , respectively. Thus, $L_T^0 = L$. We also denote $N = N_K$. Then the group A has a *polar decomposition*; i.e., each $a \in A$ can be written uniquely as

$$(1) \quad a = r(\xi^{1/2})l,$$

where $\xi \in \Omega$, $\xi^{1/2}$ is its unique root in Ω , $l \in L$, and $r(\cdot)$ is given by (3.8.5). Relative to this polar decomposition, the Haar measure on A decomposes as

$$(2) \quad da = d_* \xi dl$$

where $d_* \xi = \Delta(\xi)^{-\mu} d\xi$ is an A -invariant measure on the cone Ω and dl is the normalized Haar measure on L .

Related to the polar decomposition (1) is the *Iwasawa decomposition* $A = H N L$, where $H = \exp(i\mathfrak{h}^{-1})$ and N are the analytic subgroups having Lie algebras $i\mathfrak{h}^{-1}$ and \mathfrak{n}_K , respectively, given in Section 3.3. Each $a \in A$ has a unique decomposition $a = h n l$ with (h, n, l) in $H \times N \times L$. The group N is nilpotent, H is abelian, and H normalizes N . The decomposition of Haar measure on A with respect to this decomposition is

$$(3) \quad da = dh dn dl$$

where dh , dn and dl denote Haar measures on H , N , and L , respectively.

Relative to a choice of basis, the root spaces \mathfrak{n}_K^{jk} consist of vectors $q_{jk} \in \mathbb{R}^\nu$. Since the exponential map is a bijection of \mathfrak{n}_K with N , the coordinates $q_{jk} \in \mathbb{R}^\nu$ for $1 \leq j < k \leq r$ parametrize N . Similarly, the mapping $(q_1, \dots, q_r) \mapsto h = \exp\left(\sum_{j=1}^r (\log q_j) h_j\right)$ is an isomorphism of the multiplicative group $(\mathbb{R}^+)^r$ with H . In these coordinates,

$$(4) \quad dh = \prod_{i=1}^r \frac{dq_i}{q_i} \quad \text{and} \quad dn = dn(q) = \prod_{j=1}^r \prod_{k=1}^{j-1} dq_{jk},$$

respectively. The group $B = H N$ acts simply transitively on Ω , which implies that the mapping $b \mapsto x = b \cdot e$ is a diffeomorphism of B with Ω . Then

$$(5) \quad \Delta(x) = \Delta(h \cdot e) = \prod_{j=1}^r q_j^2 \quad \text{and} \quad (x|e) = (b \cdot e|e) = \sum_{j=1}^r q_j^2 \left(1 + \sum_{k=1}^{j-1} \|q_{jk}\|^2\right)$$

where $\|\cdot\|$ denotes the usual Euclidean norm in \mathbb{R}^ν .

4. WEIGHTED LAPLACE TRANSFORMS

In this section, we define the operator-valued gamma functions for a Siegel domain Σ of type II by an integral analogous to the integral formula for the classical gamma function, and index them by irreducible representations π of the group $K^\mathbb{C}$. We study a weighted Laplace transform on $\Omega \oplus V_1$, its inversion formula, and the Plancherel formula for this transform.

4.1. The restricted lowest weight. Let π be an irreducible finite-dimensional holomorphic (ifdh) representation of $K^\mathbb{C}$, $\mathcal{V} = \mathcal{V}_\pi$ be the space on which π acts, and denote by $\mathcal{E} = \mathcal{E}_\pi$ the space of linear transformations on \mathcal{V}_π . Endow \mathcal{E} with the inner product

$$(1) \quad (S|T) = \text{tr}(T^* S).$$

The representation π is characterized by its lowest weight, and the restriction of the lowest weight to $i\mathfrak{h}^{-1}$ is determined by its signature. The signature is an r -tuple $M = (m_1, m_2, \dots, m_r)$ of integers such that

$$(2) \quad m_1 \geq m_2 \geq \dots \geq m_r.$$

At the Lie algebra level, the signature corresponds to the linear functional

$$(3) \quad \Lambda_M = -\frac{1}{2} \sum_{j=1}^r m_j \gamma_j,$$

which is called the restricted lowest weight of π .

The lowest weight becomes a character χ_M of H , called the lowest weight character of π on H . If we use the coordinates and write $h = \exp(\sum_{j=1}^r (\log q_j) h_j)$, then

$$(4) \quad \chi_M(h) = q_1^{-m_1} q_2^{-m_2} \cdots q_r^{-m_r}.$$

We designate the last (and smallest) entry in the signature by the notation

$$(5) \quad \omega(\pi) = m_r.$$

4.2. Notation. By the proof of Proposition 3.9, if $z \in V_2$ is invertible, then $r(z) \in K_T^{\mathbb{C}}$. Thus, $\pi(r(z))$ is defined for all invertible $z \in V_2$. For convenience we adopt the notation $\pi(z)$ rather than the more cumbersome $\pi(r(z))$; i.e., by definition,

$$(1) \quad \pi(z) = \pi(r(z))$$

for all invertible elements $z \in V_2$. In particular, $\pi(\xi)$ is defined for all $\xi \in \Omega$ and $\pi(z)$ is defined for all $z \in V_2$ with $\text{Im } z > 0$.

Note that A normalizes $K_{-\frac{1}{2}}$, and $AK_{-\frac{1}{2}}$ is a subgroup of $K^{\mathbb{C}}$.

4.3. Definition. Let Δ be defined by (3.4.6). For $\alpha \in \mathbb{C}$ and $(z, u) \in \Sigma$, set

$$(1) \quad \begin{aligned} \Gamma((z, u), \alpha, \pi) &= \int_{AK_{-\frac{1}{2}}} e^{2i(az|e)} e^{2(au|u_0)} e^{-(u_0|u_0)} \\ &\times \pi((a k(i u_0))^* a k(i u_0))^{-1} \Delta(a e)^{\alpha} da du_0. \end{aligned}$$

Whenever the integral (1) converges absolutely, the mapping $((z, u), \alpha, \pi) \mapsto \Gamma((z, u), \alpha, \pi)$ is called the (generalized) gamma function for the domain Σ .

In general, the gamma function is operator-valued; that is, $\Gamma((z, u), \alpha, \pi) \in \mathcal{E}_{\pi}$. When $(z, u) = (\frac{ie}{2}, 0)$, we set

$$(2) \quad \begin{aligned} \Gamma_{\pi}(\alpha) &= \Gamma((\frac{ie}{2}, 0), \alpha, \pi) \\ &= \int_{AK_{-\frac{1}{2}}} e^{-(ae|e)} e^{-(u_0|u_0)} \pi((a k(i u_0))^* a k(i u_0))^{-1} \Delta(a e)^{\alpha} da du_0 \end{aligned}$$

and refer to the \mathcal{E}_{π} -valued mapping $\alpha \mapsto \Gamma_{\pi}(\alpha)$ as the gamma function for Ω of weight π .

By (3.10.1), we write $a = r(\xi^{\frac{1}{2}})l$, and then $a a^* = r(\xi)$. By (3.10.2) we can write (2) as

$$(3) \quad \Gamma_{\pi}(\alpha) = \int_{\Omega \oplus V_1} e^{-(\xi|e)} e^{-(u_0|u_0)} \pi(k(i u_0)^* r(\xi) k(i u_0))^{-1} \Delta(\xi)^{\alpha-\mu} d\xi du_0.$$

4.4. Theorem. Let $\alpha \in \mathbb{C}$ and $(z, u) \in \Sigma$. Then the integral defining $\Gamma((z, u), \alpha, \pi)$ converges absolutely if and only if

$$(1) \quad \omega(\pi) > \mu - \text{Re } \alpha - 1,$$

where μ is given by (3.4.3).

Proof. Without loss of generality, we assume α is real. By (4.1.1)

$$\mathrm{tr}(\pi((a k(i u_0))^* a k(i u_0))) = \|\pi(a k(i u_0))\|^2.$$

By consideration of eigenvalues

$$\begin{aligned} \|\pi((a k(i u_0))^* a k(i u_0))\| &\leq \mathrm{tr}(\pi((a k(i u_0))^* a k(i u_0))) \\ &\leq d^{1/2} \|\pi((a k(i u_0))^* a k(i u_0))\|, \end{aligned}$$

where d is the dimension of \mathcal{V}_π . Hence, it suffices to consider the convergence of

$$(2) \quad \int_{AK_{-\frac{1}{2}}} |e^{2i(a z|e)} e^{2(a u|u_0)} e^{-(u_0|u_0)}| \|\pi(a k(i u_0))^{-1}\|^2 \Delta(a e)^\alpha da du_0.$$

Since $(z, u) \in \Sigma$, $t = \mathrm{Im} z - \frac{1}{2}B(u, \bar{u}) > 0$ (cf. (2.7.1)). By [L1], Sections 3.4 and 3.5,

$$(3) \quad (u|v) = (B(u, \bar{v})|e)$$

for $u, v \in V_1$. By (3) and (3.6.5),

$$(4) \quad (a u|a u) = (B(a u, \overline{a u})|e) = (a B(u, \bar{u})|e).$$

Then (2) is equal to

$$\begin{aligned} (5) \quad &\int_{AK_{-\frac{1}{2}}} e^{-2(at|e)} e^{-(aB(u, \bar{u})|e)} |e^{2(au|u_0)}| e^{-(u_0|u_0)} \|\pi(a k(i u_0))^{-1}\|^2 \Delta(a e)^\alpha da du_0 \\ &= \int_{AK_{-\frac{1}{2}}} e^{-2(t|a^* e)} e^{-(u_0 - au|u_0 - au)} \|\pi(a k(i u_0))^{-1}\|^2 \Delta(a e)^\alpha da du_0 \end{aligned}$$

or

$$(6) \quad \int_{K_{-\frac{1}{2}}} \left[\int_A e^{-2(t|a^* e)} \|\pi(a k(i(u_0 + au)))^{-1}\|^2 \Delta(a e)^\alpha da \right] e^{-(u_0|u_0)} du_0.$$

We first assume that (5) is convergent, then so is the integral

$$(7) \quad \int_{K_{-\frac{1}{2}}} \left[\int_A e^{-2(t|a^* e)} e^{-2(au|au)} \|\pi(a k(i u_0))^{-1} f\|^2 \Delta(a e)^\alpha da \right] e^{-2(u_0|u_0)} du_0$$

for $f \in \mathcal{V}_\pi$, since $(u_0 - au|u_0 - au) \leq 2(u_0|u_0) + 2(au|au)$. Let f be a unit lowest weight vector of π corresponding to the lowest weight $-\frac{1}{2} \sum_{j=1}^r m_j \gamma_j$, then

$$\|\pi(a k(i u_0))^{-1} f\| = \|\pi(a)^{-1} f\|.$$

By (7) and the Fubini theorem,

$$(8) \quad \int_A e^{-2(t|a^* e)} e^{-2(au|au)} \|\pi(a)^{-1} f\|^2 \Delta(a e)^\alpha da$$

is convergent.

By (3.10.4), (3.10.5) and change of variables $q_{jk} \mapsto q_j^{-1} q_{jk}$, we obtain

$$\begin{aligned} \int_N e^{-(hn|e|)} dn &= \prod_{j=1}^r \prod_{k=1}^{j-1} \left(\int_{\mathbb{R}^\nu} e^{-q_j^2(1+\|q_{jk}\|^2)} dq_{jk} \right) \\ &= \prod_{j=1}^r \left(e^{-q_j^2} \int_{\mathbb{R}^\nu} \prod_{k=1}^{j-1} e^{-\|q_j q_{jk}\|^2} dq_{jk} \right) \\ (9) \qquad &= c \eta(q) e^{-\|q\|^2} \end{aligned}$$

where $h = h(q) = h(q_1, \dots, q_r)$, c is the constant

$$(10) \qquad c = \pi^{r(r-1)\nu/4} = \prod_{k < j} \int_{\mathbb{R}^\nu} e^{-\|q_{jk}\|^2} dq_{jk},$$

and η is the Jacobian

$$(11) \qquad \eta(q) = (q_2 q_3^2 \cdots q_r^{r-1})^{-\nu}.$$

By (4), the integral (8) is equal to

$$\int_A e^{-2(a(t+B(u,\bar{u}))|e|)} \|\pi(a)^{-1} f\|^2 \Delta(ae)^\alpha da.$$

Since it is convergent, so is the integral

$$(12) \qquad \int_A e^{-(ae|e|)} \|\pi(a)^{-1} f\|^2 \Delta(ae)^\alpha da.$$

By (3.10.3), (3.10.4), (3.10.5), (4.1.4), (9), (10) and (11), it is equal to

$$\begin{aligned} &c \int_H e^{-\|q\|^2} \eta(q) \chi_M(h(q))^{-2} \Delta(h(q)e)^\alpha dh(q) \\ &= c \prod_{j=1}^r \int_0^\infty e^{-q_j^2} q_j^{2(m_j+\alpha)-\nu(j-1)-1} dq_j \\ &= c 2^{-r} \prod_{j=1}^r \Gamma(m_j + \alpha - (j-1)\nu/2). \end{aligned}$$

Since it is convergent, $m_r + \alpha - (r-1)\nu/2 > 0$ and (1) follows from (3.4.4).

Conversely, if (1) is valid, then the restriction of π to A is generally reducible. By Cartan's Theorem for the lowest weight [Kn], Proposition 4.34, any irreducible component π' of this restriction has the lowest weight

$$(13) \qquad \Lambda_b = -\frac{1}{2} \sum_{j=1}^r b_j \gamma_j,$$

and $b_r \geq m_r = \omega(\pi)$. Then

$$b_r > \mu - \operatorname{Re} \alpha - 1,$$

and by the same calculation as one in the last paragraph (cf. [DG1], Theorem 2.5)

$$\int_A e^{-2(t|a^*e|)} \|\pi'(a)^{-1}\|^2 \Delta(ae)^\alpha da < \infty,$$

and hence

$$\int_A e^{-2(t|a^*e)} \|\pi(a)^{-1}\|^2 \Delta(ae)^\alpha da < \infty.$$

By [RV1, Proposition 5.6(b)], $\pi(k(iu))$ is a polynomial in u . Then the integral

$$\int_A e^{-2(t|a^*e)} \|\pi(ak(i(u_0 + au)))^{-1}\|^2 \Delta(ae)^\alpha da,$$

and the integral (6) are also convergent. The theorem is now proved.

4.5. Corollary. *Let π be an ifdh representation of $K^\mathbb{C}$ and α be a real number such that*

$$(1) \quad \omega(\pi) > \mu - \alpha - 1.$$

Then $\Gamma_\pi(\alpha)$ is a positive definite linear transformation on \mathcal{V}_π . Moreover,

$$(2) \quad \pi(l)\Gamma_\pi(\alpha) = \Gamma_\pi(\alpha)\pi(l)$$

for all $l \in L$.

4.6. Weighted Laplace transforms. As in Section 4.1, let π be an ifdh representation of $K^\mathbb{C}$, let \mathcal{V}_π be the space of π , and let \mathcal{E} be the space of linear transformations on \mathcal{V}_π . Let $\phi : \Omega \oplus V_1 \rightarrow \mathcal{E}$ be a measurable function such that it is holomorphic in V_1 and the integral

$$(1) \quad \Phi(z, u) = \pi^{-\frac{n}{2}} \int_{\Omega \oplus V_1} e^{i(z|\xi)} e^{(\xi^{\frac{1}{2}} \circ u | u_0)} e^{-(u_0 | u_0)} \pi(r(\xi^{\frac{1}{2}}) k(iu_0))^*{}^{-1} \phi(\xi, u_0) d\xi du_0$$

converges absolutely for all $(z, u) \in \Sigma$. We call the function Φ the π -weighted Laplace transform of ϕ .

By the definition (2.7.1) of Σ , any element in Σ can be expressed by $(x + iy, u)$, where $y = t + \frac{1}{2}B(u, \bar{u})$ with $t > 0$.

4.7. Proposition. *Whenever the function $x \mapsto \Phi(x + iy, u)$ is integrable,*

$$\begin{aligned} \phi(\xi, u_0) &= 2^{-n_1} \pi^{-(\frac{n}{2} + rb)} \int_{J \oplus V_1} e^{-i(x + i(t + \frac{1}{2}B(u, \bar{u}))|\xi)} e^{(u_0|\xi^{\frac{1}{2}} \circ u)} e^{-(\xi^{\frac{1}{2}} \circ u | \xi^{\frac{1}{2}} \circ u)} \\ (1) \quad &\times \Delta(\xi)^b \pi(r(\xi^{\frac{1}{2}}) k(i\xi^{\frac{1}{2}} \circ u))^* \Phi(x + i(t + \frac{1}{2}B(u, \bar{u})), u) dx du, \end{aligned}$$

where the integral is independent of the choice of $t > 0$. The formula (1) is called the inverse formula of the π -weighted Laplace transform (4.6.1).

Proof. By (4.6.1)

$$\begin{aligned} &2^{-n_1} \pi^{-(\frac{n}{2} + rb)} \int_{J \oplus V_1} e^{-i(x + i(t + \frac{1}{2}B(u, \bar{u}))|\xi)} e^{(u_0|\xi^{\frac{1}{2}} \circ u)} e^{-(\xi^{\frac{1}{2}} \circ u | \xi^{\frac{1}{2}} \circ u)} \\ &\times \Delta(\xi)^b \pi(r(\xi^{\frac{1}{2}}) k(i\xi^{\frac{1}{2}} \circ u))^* \Phi(x + i(t + \frac{1}{2}B(u, \bar{u})), u) dx du \\ &= 2^{-n_1} \pi^{-(n + rb)} \int_{J \oplus V_1} e^{-i(x + i(t + \frac{1}{2}B(u, \bar{u}))|\xi)} e^{(u_0|\xi^{\frac{1}{2}} \circ u)} e^{-(\xi^{\frac{1}{2}} \circ u | \xi^{\frac{1}{2}} \circ u)} \\ &\times \Delta(\xi)^b \pi(r(\xi^{\frac{1}{2}}) k(i\xi^{\frac{1}{2}} \circ u))^* \left[\int_{\Omega \oplus V_1} e^{i(x + i(t + \frac{1}{2}B(u, \bar{u}))|\xi')} \right. \\ (2) \quad &\times e^{(\xi'^{\frac{1}{2}} \circ u | u'_0)} e^{-(u'_0 | u'_0)} \pi(r(\xi'^{\frac{1}{2}}) k(iu'_0))^*{}^{-1} \phi(\xi', u'_0) d\xi' du'_0 \Big] dx du. \end{aligned}$$

By the classical inverse Laplace transform formula (cf. [RV1]),

$$\begin{aligned} & (2\pi)^{-n_1} \int_J e^{-i(x+i(t+\frac{1}{2}B(u,\bar{u}))|\xi)} \left[\int_{\Omega} e^{i(x+i(t+\frac{1}{2}B(u,\bar{u}))|\xi')} \right. \\ & \quad \times e^{(\xi' \frac{1}{2} \circ u | u'_0)} \pi(r(\xi' \frac{1}{2}) k(i u'_0))^*{}^{-1} \phi(\xi', u'_0) d\xi' \Big] dx \\ & = e^{(\xi \frac{1}{2} \circ u | u'_0)} \pi(r(\xi \frac{1}{2}) k(i u'_0))^*{}^{-1} \phi(\xi, u'_0). \end{aligned}$$

Since $n = n_1 + rb$, (2) is equal to

$$\begin{aligned} & \pi^{-2rb} \int_{V_1} e^{(u_0 | \xi \frac{1}{2} \circ u)} e^{-(\xi \frac{1}{2} \circ u | \xi \frac{1}{2} \circ u)} \Delta(\xi)^b \pi(r(\xi \frac{1}{2}) k(i \xi \frac{1}{2} \circ u))^* \\ (3) \quad & \times \left[\int_{V_1} e^{(\xi \frac{1}{2} \circ u | u'_0)} e^{-(u'_0 | u'_0)} \pi(r(\xi \frac{1}{2}) k(i u'_0))^*{}^{-1} \phi(\xi, u'_0) du'_0 \right] du. \end{aligned}$$

By the reproducing kernel property of a Fock space (cf. [Di2], Section 4), if ϕ is holomorphic in a complex vector space V and integrable with respect to the measure $e^{-(u|u)} du$ of V , then

$$(4) \quad \phi(u'_0) = \pi^{-d} \int_V e^{(u'_0|u)} e^{-(u|u)} \phi(u) du,$$

where d is the dimension of V over the complex field \mathbb{C} . Note that $\dim_{\mathbb{C}} V_1 = rb$. By [Gi], (2.50), the Jacobian of the map $u \mapsto \xi \frac{1}{2} \circ u$ on V_1 is $\Delta(\xi)^b$ for $\xi \in \Omega$. By (3.9.1), $k(u)$ is holomorphic in u , so is $e^{(u|u_0)} \pi(r(\xi \frac{1}{2}) k(i u))$. Letting $\phi(u) = e^{(u|u_0)} \pi(r(\xi \frac{1}{2}) k(i u))$, changing variables $u \mapsto \xi \frac{1}{2} \circ u$ and taking adjoints on both sides of (4), we have

$$\begin{aligned} & \pi^{-rb} \int_{V_1} e^{(u_0 | \xi \frac{1}{2} \circ u)} e^{-(\xi \frac{1}{2} \circ u | \xi \frac{1}{2} \circ u)} \Delta(\xi)^b \pi(r(\xi \frac{1}{2}) k(i \xi \frac{1}{2} \circ u))^* e^{(\xi \frac{1}{2} \circ u | u'_0)} du \\ & = e^{(u_0 | u'_0)} \pi(r(\xi \frac{1}{2}) k(i u'_0))^*. \end{aligned}$$

Then by (4) again, (3) is equal to

$$\pi^{-rb} \int_{V_1} e^{-(u'_0 | u'_0)} e^{(u_0 | u'_0)} \phi(\xi, u'_0) du'_0 = \phi(\xi, u_0).$$

and (1) is proved.

We now prove a Plancherel-type theorem for the Laplace transform given by (4.6.1).

4.8. Theorem. *Let π, \mathcal{E} be given in Section 4.1, and α be a real number such that $\omega(\pi) + \alpha > \mu - 1$. Suppose $\phi : \Omega \oplus V_1 \rightarrow \mathcal{E}$ is a measurable function such that it is holomorphic in V_1 and the π -weighted Laplace transform Φ given by (4.6.1) is absolutely convergent for all $(z, u) \in \Sigma$. Then Φ is holomorphic in Σ and*

$$\begin{aligned} (1) \quad & \int_{\Sigma} \|\pi(r(2y - B(u, \bar{u})) \frac{1}{2} k(-i u))^*{}^{-1} \Phi(x + iy, u)\|^2 \Delta(2y - F(u, \bar{u}))^{-\mu+\alpha} dx dy du \\ & = \int_{\Omega \oplus V_1} (\Gamma_{\pi}(\alpha) \phi(\xi, u_0) | \phi(\xi, u_0)) \Delta(\xi)^{-\alpha-b} e^{-(u_0 | u_0)} d\xi du_0. \end{aligned}$$

Proof. That Φ is holomorphic in Σ follows from the theory of the Laplace transform. By (3.9.1), (3.6.5) and a computation,

$$(2) \quad k(i u_0) a = a k(i a^* u_0).$$

By (4.6.1), and the Plancherel formula for the functions on J , we have

$$\begin{aligned} & \int_J \|\pi(r(2t)^{\frac{1}{2}} k(-i u))^* \Phi(x + i(t + \frac{1}{2}B(u, \bar{u})), u)\|^2 dx \\ &= \pi^{-n} \int_J \left\| \int_{\Omega \oplus V_1} e^{i(x|\xi)} e^{-(t|\xi)} e^{-\frac{1}{2}(B(u, \bar{u})|\xi)} e^{(\xi^{\frac{1}{2}} \circ u|u_0)} e^{-(u_0|u_0)} \right. \\ & \quad \times \pi(r(2t)^{\frac{1}{2}} k(-i u))^* \pi(r(\xi^{\frac{1}{2}}) k(i u_0))^* \phi(\xi, u_0) d\xi du_0 \|^2 dx \\ &= 2^{n_1} \pi^{-rb} \int_{\Omega} \left\| \int_{V_1} e^{-(t|\xi)} e^{-\frac{1}{2}(B(u, \bar{u})|\xi)} e^{(\xi^{\frac{1}{2}} \circ u|u_0)} e^{-(u_0|u_0)} \right. \\ & \quad \times \pi(r(2t)^{\frac{1}{2}} k(-i u) r(\xi^{\frac{1}{2}}) k(i u_0))^* \phi(\xi, u_0) du_0 \|^2 d\xi \\ &= 2^{n_1} \pi^{-rb} \int_{\Omega} \left\| \int_{V_1} e^{-(t|\xi)} e^{-\frac{1}{2}(B(u, \bar{u})|\xi)} e^{(\xi^{\frac{1}{2}} \circ u|u_0)} e^{-(u_0|u_0)} \right. \\ & \quad \times \pi(r(2t)^{\frac{1}{2}} r(\xi^{\frac{1}{2}}) k(-i \xi^{\frac{1}{2}} \circ u) k(i u_0))^* \phi(\xi, u_0) du_0 \|^2 d\xi. \end{aligned}$$

Then

$$\begin{aligned} & \int_{\Sigma} \|\pi(r(2y - B(u, \bar{u}))^{\frac{1}{2}} k(-i u))^* \Phi(x + iy, u)\|^2 \Delta(2y - B(u, \bar{u}))^{-\mu+\alpha} dx dy du \\ &= 2^{n_1} \pi^{-rb} \int_{\Omega} \int_{\Omega} \int_{V_1} e^{-2(t|\xi)} \Delta(2t)^{-\mu+\alpha} \left\| \int_{V_1} e^{-\frac{1}{2}(B(u, \bar{u})|\xi)} e^{(\xi^{\frac{1}{2}} \circ u|u_0)} \right. \\ & \quad \times e^{-(u_0|u_0)} (\pi(2t)^{\frac{1}{2}} \pi(\xi^{\frac{1}{2}}) \pi(k(i(u_0 - \xi^{\frac{1}{2}} \circ u))))^* \phi(\xi, u_0) du_0 \|^2 d\xi dt du. \end{aligned}$$

By (4.4.3) and (2.7.4),

$$(3) \quad (\xi^{\frac{1}{2}} \circ u | \xi^{\frac{1}{2}} \circ u) = (B(\xi^{\frac{1}{2}} \circ u, \overline{\xi^{\frac{1}{2}} \circ u})|e) = (P(\xi^{\frac{1}{2}}) B(u, \bar{u})|e) = (B(u, \bar{u})|\xi).$$

By a change of variables and [Di2], Theorem 4.3, we have

$$\begin{aligned} & (4) \quad \int_{\Sigma} \|\pi(r(2y - B(u, \bar{u}))^{\frac{1}{2}} k(-i u))^* \Phi(x + iy, u)\|^2 \Delta(2y - B(u, \bar{u}))^{-\mu+\alpha} dx dy du \\ &= 2^{n_1} \int_{\Omega \oplus V_1} \text{tr} \left[\phi(\xi, u_0)^* \left(\int_{\Omega \oplus V_1} e^{-2(t|\xi)} e^{-(\xi^{\frac{1}{2}} \circ u | \xi^{\frac{1}{2}} \circ u)} \right. \right. \\ & \quad \times (\pi(2t)^{\frac{1}{2}} \pi(\xi^{\frac{1}{2}}) \pi(k(i \xi^{\frac{1}{2}} \circ u)))^{-1} (\pi(2t)^{\frac{1}{2}} \pi(\xi^{\frac{1}{2}}) \pi(k(i \xi^{\frac{1}{2}} \circ u)))^* \\ & \quad \times \Delta(2t)^{-\mu+\alpha} dt du \Big) \phi(\xi, u_0) \Big] e^{-(u_0|u_0)} d\xi du_0. \end{aligned}$$

By (3.10.1), (3.10.2), (4.3.3) and variable changes, we obtain that

$$\begin{aligned} & \int_{\Omega \oplus V_1} e^{-2(t|\xi)} e^{-(\xi^{\frac{1}{2}} \circ u | \xi^{\frac{1}{2}} \circ u)} (\pi(2t)^{\frac{1}{2}} \pi(\xi^{\frac{1}{2}}) \pi(k(i \xi^{\frac{1}{2}} \circ u)))^{-1} \\ & \quad \times (\pi(2t)^{\frac{1}{2}} \pi(\xi^{\frac{1}{2}}) \pi(k(i \xi^{\frac{1}{2}} \circ u)))^* \Delta(2t)^{-\mu+\alpha} dt du \\ (5) \quad &= 2^{-n_1} \Delta(\xi)^{-\alpha-b} \Gamma_{\pi}(\alpha). \end{aligned}$$

Now (1) follows directly from (4) and (5).

4.9. Corollary. Let π , \mathcal{E} be given in Section 4.1, and let α be a real number such that $\omega(\pi) > \mu - \alpha - 1$. Suppose that $\phi : AK_{-\frac{1}{2}} \rightarrow \mathcal{E}$ is a measurable function such that it is holomorphic in u_0 , it satisfies the π -covariant condition

$$(1) \quad \phi(a k(u_0) l) = \pi(l)^{-1} \phi(a k(u_0))$$

for $a \in A$, $u_0 \in V_1$, $l \in L$, and the integral

$$(2) \quad \begin{aligned} \Phi(z, u) = & \pi^{-\frac{n}{2}} \int_{AK_{-\frac{1}{2}}} e^{i(z|ae)} e^{(u|au_0)} e^{-(u_0|u_0)} \\ & \times \pi(a k(i u_0))^{*-1} \phi(a k(u_0)) \Delta(a e)^\mu da du_0 \end{aligned}$$

is convergent absolutely for all $(z, u) \in \Sigma$. Then Φ is holomorphic in Σ and

$$(3) \quad \begin{aligned} & \int_{\Sigma} \|\pi(r(2y - B(u, \bar{u}))^{\frac{1}{2}} k(-i u))^{*-1} \Phi(x + iy, u)\|^2 \Delta(2y - B(u, \bar{u}))^{-\mu+\alpha} dx dy du \\ & = \int_{AK_{-\frac{1}{2}}} (\Gamma_{\pi}(\alpha) \phi(a k(u_0)) | \phi(a k(u_0)) \Delta(a e)^{\mu-\alpha-b} e^{-(u_0|u_0)} da du_0. \end{aligned}$$

(Note that we do not assert the finiteness of the two integrals in (3); however, if one integral is infinite, then so is the other. The same is true for (4.8.1). Note that Theorem 4.8 and Corollary 4.9 are also valid for \mathcal{V}_{π} -valued functions ϕ and Φ .)

Proof. The Corollary follows from Theorem 4.8, (3.10.1) and (3.10.2).

5. OPERATOR-VALUED BESSEL FUNCTIONS

In this section, we define the operator-valued Bessel function by the inverse Laplace transform, and study its square-integrability with respect to certain measures by Theorem 4.8.

5.1. Inversion formula on the cone. Let $\phi : \Omega \rightarrow \mathcal{E}$ be a measurable function on the cone Ω such that

$$(1) \quad \Phi(x + iy) = (2\pi)^{-\frac{n+1}{2}} \int_{\Omega} e^{i(x+iy|\xi)} \pi(\xi^{\frac{1}{2}})^{-1} \phi(\xi) d\xi$$

converges absolutely for all $x + iy \in J + i\Omega$. We call Φ the π -weighted Laplace transform of ϕ . The Laplace inversion formula

$$(2) \quad \phi(\xi) = (2\pi)^{-\frac{n+1}{2}} \int_J e^{-i(x+iy|\xi)} \pi(\xi^{\frac{1}{2}}) \Phi(x + iy) dx$$

is valid whenever the function $x \mapsto \Phi(x + iy)$ is integrable. Conversely, suppose $\Phi : J + i\Omega \rightarrow \mathcal{E}$ is holomorphic, the integral

$$(3) \quad I_{\Phi}(y) = \int_J \|\Phi(x + iy)\| dx$$

exists for all $y \in \Omega$, and as a function of y the integral I_{Φ} is bounded on any compact subset of Ω . Then there exists a continuous function ϕ on J such that Φ is the π -weighted Laplace transform of ϕ defined by (1); and if for any $\epsilon > 0$ the function $y \mapsto I_{\Phi}(y) e^{-\epsilon\|y\|}$ is bounded as $\|y\| \rightarrow \infty$, then $\phi(\xi) = 0$ for all ξ not in Ω . In fact, ϕ is determined by (2) (see [B2]).

5.2. Lemma. *If $\omega(\pi) > 2\mu - 1$, then*

$$(1) \quad \beta = \int_J [\operatorname{tr}(\pi(x^2 + e))]^{1/2} dx$$

is finite ([DG1], Lemma 3.3), and

$$(2) \quad \int_J \|\pi(x + iy)\| dx \leq \Delta(y)^\mu \|\pi(y^{\frac{1}{2}})\|^2 \beta.$$

for $y \in \Omega$ ([DG1], Section 3.4).

5.3. Theorem. *Suppose that $\omega(\pi) > 2\mu - 1$. For fixed $v_0 \in V_1$ the integral*

$$(1) \quad 2^{-n_1} \pi^{-\left(\frac{n}{2} + rb\right)} \int_{J \oplus V_1} e^{-i(x + i(t + \frac{1}{2}B(u, \bar{u}))|\xi)} e^{(u_0|\xi^{\frac{1}{2}} \circ u)} e^{-(\xi^{\frac{1}{2}} \circ u|\xi^{\frac{1}{2}} \circ u)} \\ \times e^{-i((x + i(t + \frac{1}{2}B(u, \bar{u})))^{-1}|e)} e^{((x + i(t + \frac{1}{2}B(u, \bar{u})))^{-1}|B(v_0, \bar{u}))} \Delta(\xi)^b \\ \times \pi(r(\xi^{\frac{1}{2}})k(i\xi^{\frac{1}{2}} \circ u)) \pi((x + i(t + \frac{1}{2}B(u, \bar{u}))) \pi(k(u)) \pi(k(-i v_0))^* dx du$$

converges absolutely, is independent of the choice of $t \in \Omega$, and defines an \mathcal{E} -valued continuous function $\phi(\xi, u_0)$ on $\Omega \oplus V_1$. Moreover, ϕ is holomorphic in u_0 and the π -weighted Laplace transform Φ of ϕ defined by (4.6.1) is

$$(2) \quad \Phi(z, u) = e^{-i(z^{-1}|e)} e^{(z^{-1}|B(v_0, \bar{u}))} \pi(z) \pi(k(u)) \pi(k(-i v_0))^*,$$

where $(z, u) \in \Sigma$.

Proof. For $y \in \Omega$, $\operatorname{Im}[(x + iy)^{-1}] < 0$. Then $\operatorname{Re}[-i((x + iy)^{-1}|e)] < 0$ and $|e^{-i(x + iy)^{-1}|e}| < 1$. Since $B(\cdot, \cdot)$ is positive definite, $B(u - v_0, \bar{u} - \bar{v}_0) \geq 0$ and $B(u + v_0, \bar{u} + \bar{v}_0) \geq 0$. Hence,

$$-\frac{1}{2}[B(u, \bar{u}) + B(v_0, \bar{v}_0)] \leq \operatorname{Re} B(v_0, \bar{u}) \leq \frac{1}{2}[B(u, \bar{u}) + B(v_0, \bar{v}_0)].$$

Replacing v_0 by iv_0 , we have

$$-\frac{1}{2}[B(u, \bar{u}) + B(v_0, \bar{v}_0)] \leq \operatorname{Im} B(v_0, \bar{u}) \leq \frac{1}{2}[B(u, \bar{u}) + B(v_0, \bar{v}_0)].$$

Thus,

$$|((x + i(t + \frac{1}{2}B(u, \bar{u})))^{-1}|B(v_0, \bar{u}))| \\ \leq |((x + i(t + \frac{1}{2}B(u, \bar{u})))^{-1}|B(u, \bar{u}))| + |((x + i(t + \frac{1}{2}B(u, \bar{u})))^{-1}|B(v_0, \bar{v}_0))|.$$

$|((x + i(t + \frac{1}{2}B(u, \bar{u})))^{-1}|B(v_0, \bar{u}))|$ and $|e^{((x + i(t + \frac{1}{2}B(u, \bar{u})))^{-1}|B(v_0, \bar{u}))}|$ are bounded for fixed $t \in \Omega$ and $v_0 \in V_1$. By Section 5.1 and Lemma 5.2, the integral

$$\int_J e^{-i(x + i(t + \frac{1}{2}B(u, \bar{u}))|\xi)} e^{-i((x + i(t + \frac{1}{2}B(u, \bar{u})))^{-1}|e)} \\ \times e^{((x + i(t + \frac{1}{2}B(u, \bar{u})))^{-1}|B(v_0, \bar{u}))} \pi(x + i(t + \frac{1}{2}B(u, \bar{u}))) dx$$

converges absolutely and the integral vanishes for all ξ not in Ω . Since $\pi(k(u))$ is a polynomial in u (cf. [RV1], Proposition 5.6(b)), the integral (1) also converges absolutely, and it defines an \mathcal{E} -valued function $\phi(\xi, u_0)$.

By (1), (4.6.1) and (4.7.4)

$$\begin{aligned}
\Phi(z, u) &= 2^{-n_1} \pi^{-(n+rb)} \int_{\Omega \oplus V_1} e^{i(z|\xi)} e^{(\xi^{\frac{1}{2}} \circ u | u_0)} e^{-(u_0 | u_0)} \pi(r(\xi^{\frac{1}{2}}) k(i u_0))^*{}^{-1} \\
&\times \left[\int_{J \oplus V_1} e^{-i(x' + iy' | \xi)} e^{(u_0 | \xi^{\frac{1}{2}} \circ u')} e^{-(\xi^{\frac{1}{2}} \circ u' | \xi^{\frac{1}{2}} \circ u')} e^{-i((x' + iy')^{-1} | e)} e^{((x' + iy')^{-1} | B(v_0, \bar{u}'))} \right. \\
&\quad \times \Delta(\xi)^b \pi(r(\xi^{\frac{1}{2}}) k(i \xi^{\frac{1}{2}} \circ u'))^* \pi(x' + iy') \pi(k(u')) \pi(k(-i v_0))^* dx' du' \Big] d\xi du_0 \\
&= 2^{-n_1} \pi^{-n} \int_{\Omega} e^{i(z|\xi)} \left[\int_{J \oplus V_1} e^{(\xi^{\frac{1}{2}} \circ u | \xi^{\frac{1}{2}} \circ u')} e^{-i(x' + iy' | \xi)} e^{-(\xi^{\frac{1}{2}} \circ u' | \xi^{\frac{1}{2}} \circ u')} \right. \\
&\times e^{-i((x' + iy')^{-1} | e)} e^{((x' + iy')^{-1} | B(v_0, \bar{u}'))} \Delta(\xi)^b \pi(r(\xi^{\frac{1}{2}}) k(i \xi^{\frac{1}{2}} \circ u))^*{}^{-1} \\
&\quad \times \pi(r(\xi^{\frac{1}{2}}) k(i \xi^{\frac{1}{2}} \circ u'))^* \pi(x' + iy') \pi(k(u')) \pi(k(-i v_0))^* dx' du' \Big] d\xi \\
&= 2^{-n_1} \pi^{-n_1} \int_{\Omega} e^{i(z|\xi)} \left[\int_J e^{-i(x' + iy' | \xi)} e^{-i((x' + iy')^{-1} | e)} \right. \\
&\quad \times e^{((x' + iy')^{-1} | B(v_0, \bar{u}))} \pi(x' + iy') \pi(k(u)) \pi(k(-i v_0))^* dx' \Big] d\xi.
\end{aligned}$$

Then (2) follows from Section 5.1 and Lemma 5.2.

We now define the operator-valued Bessel function K_π by the inverse Laplace transform.

5.4. Definition. By Theorem 5.3, there is an \mathcal{E} -valued function $K_\pi(\xi, u_0; v_0)$ on $\Omega \oplus V_1 \oplus V_1$ such that

$$\begin{aligned}
&\int_{\Omega \oplus V_1} e^{i(z|\xi)} e^{(\xi^{\frac{1}{2}} \circ u | u_0)} e^{-(u_0 | u_0)} \pi(r(\xi^{\frac{1}{2}}) k(i u_0))^*{}^{-1} K_\pi(\xi, u_0; v_0) \Delta(\xi)^{-\mu} d\xi du_0 \\
(1) \quad &= e^{-i(z^{-1} | e)} e^{(z^{-1} | B(v_0, \bar{u}))} \pi(z) \pi(k(u)) \pi(k(-i v_0))^*.
\end{aligned}$$

We call K_π the (operator-valued) Bessel function of weight π .

By polar coordinates, we can define $K_\pi(a k(u_0); v_0)$ on $(AK_{-\frac{1}{2}}) \times V_1$. By (3.10.1) and (3.10.2), we have

$$\begin{aligned}
&\int_{AK_{-\frac{1}{2}}} e^{i(z|ae)} e^{(u|au_0)} e^{-(u_0|u_0)} \pi(ak(iu_0))^*{}^{-1} K_\pi(ak(u_0); v_0) da du_0 \\
(2) \quad &= e^{-i(z^{-1}|e)} e^{(z^{-1}|B(v_0, \bar{u}))} \pi(z) \pi(k(u)) \pi(k(-iv_0))^*.
\end{aligned}$$

5.5. Lemma. For $u \in V_1$ and invertible $z \in V_2$, we have

$$\begin{aligned}
&k(-u) r(z)^{-1} k(-iu)^* r(2\text{Im } z - B(u, \bar{u})) k(-iu) r(z^*)^{-1} k(-u)^* \\
(1) \quad &= k(-iz^{-1} \circ u)^* r(2\text{Im}(-z^{-1}) - B(z^{-1} \circ u, \overline{z^{-1} \circ u})) k(-iz^{-1} \circ u).
\end{aligned}$$

Proof. By (3.9.1), (3.9.2) and (3.8.5), we have

$$\begin{aligned}
 (2) \quad & k(-u) r(z^{-1}) k(-i u)^* r(2\text{Im } z - B(u, \bar{u})) k(-i u) r(z^*)^{-1} k(-u)^* \cdot (w, v) \\
 &= \left(P(z)^{-1} P(2\text{Im } z - B(u, \bar{u})) P(z^*)^{-1} w \right. \\
 &\quad + P(z)^{-1} P(2\text{Im } z - B(u, \bar{u})) P(z^*)^{-1} B(v, -\bar{u}) \\
 &\quad + P(z)^{-1} B((2\text{Im } z - B(u, \bar{u})) \circ \{ [P(z^*)^{-1} w + P(z^*)^{-1} B(v, -\bar{u})] \circ (-i u) \\
 &\quad \quad + z^{*-1} \circ v \}, i \bar{u}), \{ P(z)^{-1} P(2\text{Im } z - B(u, \bar{u})) P(z^*)^{-1} w \\
 &\quad + P(z)^{-1} P(2\text{Im } z - B(u, \bar{u})) P(z^*)^{-1} B(v, -\bar{u}) + P(z)^{-1} B((2\text{Im } z - B(u, \bar{u})) \\
 &\quad \quad \circ ([P(z^*)^{-1} w + P(z^*)^{-1} B(v, -\bar{u})] \circ (-i u) + z^{*-1} \circ v), i \bar{u}) \} \circ (-u) \\
 &\quad \left. + z^{-1} \circ \{ (2\text{Im } z - B(u, \bar{u})) \circ ([P(z^*)^{-1} w + P(z^*)^{-1} B(v, \bar{u})] \circ (-i u) + z^{*-1} \circ v) \} \right),
 \end{aligned}$$

and

$$\begin{aligned}
 (3) \quad & k(-i z^{-1} \circ u)^* r(2\text{Im}(-z^{-1}) - B(z^{-1} \circ u, \overline{z^{-1} \circ u})) k(-i z^{-1} \circ u) \cdot (w, v) \\
 &= \left(P(-2\text{Im}(z^{-1}) - B(z^{-1} \circ u, \overline{z^{-1} \circ u})) w \right. \\
 &\quad + B((-2\text{Im}(z^{-1}) - B(z^{-1} \circ u, \overline{z^{-1} \circ u})) \circ (w \circ (-i z^{-1} \circ u) + v), \overline{-i z^{-1} \circ u}), \\
 &\quad \left. (-2\text{Im}(z^{-1}) - B(z^{-1} \circ u, \overline{z^{-1} \circ u})) \circ (w \circ (-i z^{-1} \circ u) + v) \right).
 \end{aligned}$$

for $(w, v) \in \Sigma$.

By (2.2.1), (2.7.5), (2.7.6) and a calculation, we have

$$\begin{aligned}
 & P(z)^{-1} P(2\text{Im } z - B(u, \bar{u})) P(z^*)^{-1} w \\
 &= P(z)^{-1} \{ [P(2\text{Im } z) + P(B(u, \bar{u}))] P(z^*)^{-1} w - B((2\text{Im } z) \circ (P(z^*)^{-1} w \circ u), \bar{u}) \\
 &\quad - B(u, \overline{(2\text{Im } z) \circ ((P(z^*)^{-1} w)^* \circ u)}) \}, \\
 & \quad P(-2\text{Im}(z^{-1}) - B(z^{-1} \circ u, \overline{z^{-1} \circ u})) w \\
 & \quad = P(-2\text{Im}(z^{-1})) w + P(B(z^{-1} \circ u, \overline{z^{-1} \circ u})) w \\
 & \quad - B(-2\text{Im}(z^{-1}) \circ (w \circ (z^{-1} \circ u)), \overline{z^{-1} \circ u}) \\
 & \quad - B(z^{-1} \circ u, \overline{-2\text{Im}(z^{-1}) \circ (w^* \circ (z^{-1} \circ u))}), \\
 & P(z)^{-1} P(B(u, \bar{u})) P(z^*)^{-1} w \\
 &= \frac{1}{2} P(z)^{-1} \{ B(B(u, \bar{u}) \circ (P(z^*)^{-1} w \circ u), \bar{u}) \\
 &\quad + B(u, \overline{B(u, \bar{u})^* \circ ((P(z^*)^{-1} w)^* \circ u)}) \},
 \end{aligned}$$

and

$$\begin{aligned}
 & P(B(z^{-1} \circ u, \overline{z^{-1} \circ u})) w = \frac{1}{2} \{ B(B(z^{-1} \circ u, \overline{z^{-1} \circ u}) \circ (w \circ (z^{-1} \circ u)), \overline{z^{-1} \circ u}) \\
 &\quad + B(z^{-1} \circ u, \overline{B(z^{-1} \circ u, \overline{z^{-1} \circ u})^* \circ (w^* \circ (z^{-1} \circ u))}) \}.
 \end{aligned}$$

By [Do2], (4.2) and [Do1], Theorem 3.3,

$$B(B(u, \bar{u}) \circ (P(z^*)^{-1} w \circ u), \bar{u}) = B(u, \overline{B(u, \bar{u})^* \circ ((P(z^*)^{-1} w)^* \circ u)})$$

and

$$\begin{aligned} & B(B(z^{-1} \circ u, \overline{z^{-1} \circ u}) \circ (w \circ (z^{-1} \circ u)), \overline{z^{-1} \circ u}) \\ &= B(z^{-1} \circ u, \overline{B(z^{-1} \circ u, \overline{z^{-1} \circ u})^* \circ (w^* \circ (z^{-1} \circ u))}). \end{aligned}$$

By (2.2.4), (2.7.4) and (2.7.6), we have

$$\begin{aligned} & P(z)^{-1} P(2\operatorname{Im} z - B(u, \bar{u})) P(z^*)^{-1} w \\ & \quad + P(z)^{-1} B((2\operatorname{Im} z - B(u, \bar{u})) \circ [P(z^*)^{-1} w \circ u], \bar{u}) \\ &= P(z)^{-1} P(2\operatorname{Im} z) P(z^*)^{-1} w - P(z)^{-1} B(u, \overline{(2\operatorname{Im} z) \circ ((P(z^*)^{-1} w)^* \circ u)}) \\ &= P(-2\operatorname{Im}(z^{-1})) w - B(z^{-1} \circ u, \overline{(-2\operatorname{Im}(z^{-1})) \circ (w^* \circ (z^{-1} \circ u))}) \\ &= P(-2\operatorname{Im}(z^{-1}) - B(z^{-1} \circ u, \overline{z^{-1} \circ u})) w \\ & \quad + B((-2\operatorname{Im}(z^{-1}) - B(z^{-1} \circ u, \overline{z^{-1} \circ u})) \circ (w \circ (z^{-1} \circ u)), \overline{z^{-1} \circ u}). \end{aligned}$$

By similar calculations, we can show that the right sides of (2) and (3) are equal. Hence,

$$\begin{aligned} & k(-u) r(z^{-1}) k(-i u)^* r(2\operatorname{Im} z - B(u, \bar{u})) k(-i u) r(z^*)^{-1} k(-u)^* \cdot (w, v) \\ &= k(-i z^{-1} \circ u)^* r(2\operatorname{Im}(-z^{-1}) - B(z^{-1} \circ u, \overline{z^{-1} \circ u})) k(-i z^{-1} \circ u) \cdot (w, v) \end{aligned}$$

for $(w, v) \in \Sigma$, and the lemma is proved.

We next apply Theorem 4.8 to obtain the square-integrability of K_π with respect to certain measures on $\Omega \oplus V_1 \oplus V_1$.

5.6. Theorem. *Let α be a real number such that $\alpha + \omega(\pi) > \mu - 1$. Then*

$$\begin{aligned} (1) \quad & \int_{\Omega \oplus V_1 \oplus V_1} (\Gamma_\pi(\alpha) K_\pi(\xi, u; v_0) |K_\pi(\xi, u; v_0)| \Delta(\xi)^{-2\mu-\alpha-b} e^{-(u|u)} e^{-(v_0|v_0)} d\xi du dv_0 \\ &= c \operatorname{tr} \int_{\Omega \oplus V_1} e^{-2(t|e)} e^{-(u|u)} \pi(k(i u)) \pi(2t)^{-1} \pi(k(i u))^* \Delta(2t)^{-2\mu-\alpha-b} dt du \\ & \quad \times \int_{J \oplus V_1} \Delta(x + i(e + \frac{1}{2} B(u, \bar{u})))^{-2(\mu+b+\alpha)} dx du \end{aligned}$$

for some constant c . The integrals in (1) are finite for α in the interval

$$(2) \quad -\frac{1}{2} < \alpha < \omega(\pi) - 2\mu - b + 1.$$

Proof. By Theorem 4.4, $\Gamma_\pi(\alpha)$ is well defined. We apply Theorem 4.8 with

$$\phi(\xi, u) = \pi^{\frac{n}{2}} K_\pi(\xi, u; v_0) \Delta(\xi)^{-\mu}.$$

By (5.4.1), (4.8.1), (4.6.1) and (4.1.1), we have

$$\begin{aligned}
 & \int_{\Omega \oplus V_1} (\Gamma_\pi(\alpha) K_\pi(\xi, u; v_0) | K_\pi(\xi, u; v_0)) \Delta(\xi)^{-2\mu-\alpha-b} e^{-(u|u)} e^{-(v_0|v_0)} d\xi du \\
 &= \pi^{-n} \int_{\Sigma} ||\pi(r(2y - B(u, \bar{u}))^{\frac{1}{2}} k(-i u))^* e^{-i(z^{-1}|e)} e^{(z^{-1}|B(v_0, \bar{u}))} \pi(z) \\
 &\quad \times \pi(k(u)) \pi(k(-i v_0))^* ||^2 \Delta(2y - B(u, \bar{u}))^{-\mu+\alpha} e^{-(v_0|v_0)} dx dy du \\
 &= \pi^{-n} \int_{\Sigma} |e^{-i(z^{-1}|e)}|^2 e^{2\operatorname{Re}(z^{-1}|B(v_0, \bar{u}))} e^{-(v_0|v_0)} \\
 &\quad \times \operatorname{tr} \left\{ \pi(k(-i v_0)) \pi(k(u))^* \pi(z)^* \pi(k(i u)) \pi(2y - B(u, \bar{u}))^{-1} \pi(k(i u))^* \right. \\
 &\quad \left. \times \pi(z) \pi(k(u)) \pi(k(-i v_0))^* \right\} \Delta(2y - B(u, \bar{u}))^{-\mu+\alpha} dx dy du.
 \end{aligned}
 \tag{3}$$

By (5.5.1), we have

$$\begin{aligned}
 & k(u)^* r(z^*) k(i u) r(2\operatorname{Im} z - B(u, \bar{u}))^{-1} k(i u)^* r(z) k(u) \\
 &= k(i z^{-1} \circ u) r(2\operatorname{Im}(-z^{-1}) - B(z^{-1} \circ u, \overline{z^{-1} \circ u}))^{-1} k(i z^{-1} \circ u)^*.
 \end{aligned}$$

By [Gi], (2.50), $\Delta(2y - B(u, \bar{u}))^{-2\mu-b} dx dy du$ is an invariant measure under the variable changes $z \mapsto -z^{-1}$, $u \mapsto z \circ u$. Then (3) is equal to

$$\begin{aligned}
 & \pi^{-n} \int_{\Sigma} e^{-2(y|e)} e^{2\operatorname{Re}(e|B(v_0, \bar{u}))} e^{-(v_0|v_0)} \operatorname{tr} \left\{ \pi(k(-i v_0)) \pi(k(i u)) \pi(2y - B(u, \bar{u}))^{-1} \right. \\
 &\quad \left. \times \pi(k(i u))^* \pi(k(-i v_0))^* \right\} \Delta(2y - B(u, \bar{u}))^{-\mu+\alpha} \Delta(z)^{-2(\mu+b+\alpha)} dx dy du \\
 &= \pi^{-n} \operatorname{tr} \int_{\Sigma} e^{-2(t|e)} e^{-(u-v_0|u-v_0)} \pi(k(i(u-v_0))) \pi(2t)^{-1} \\
 &\quad \times \pi(k(i(u-v_0)))^* \Delta(2t)^{-\mu+\alpha} \Delta(z)^{-2(\mu+b+\alpha)} dx dy du,
 \end{aligned}$$

where $z = x + iy$ and $t = y - \frac{1}{2}B(u, \bar{u})$. It follows that

$$\begin{aligned}
 & \int_{\Omega \oplus V_1 \oplus V_1} (\Gamma_\pi(\alpha) K_\pi(\xi, u; v_0) | K_\pi(\xi, u; v_0)) \Delta(\xi)^{-2\mu-\alpha-b} e^{-(u|u)} e^{-(v_0|v_0)} d\xi du dv_0 \\
 &= c \operatorname{tr} \int_{\Omega \oplus V_1} e^{-2(t|e)} e^{-(u|u)} \pi(k(i u)) \pi(2t)^{-1} \pi(k(i u))^* \Delta(2t)^{-2\mu-\alpha-b} dt du \\
 &\quad \times \int_{J \oplus V_1} \Delta(x + i(e + \frac{1}{2}B(u, \bar{u})))^{-2(\mu+b+\alpha)} dx du,
 \end{aligned}$$

where c is a constant. By Theorem 4.4, the first integral converges if $\omega(\pi) > 2\mu + \alpha + b - 1$. By [Gi, (5.98)], the second integral converges if $\alpha > -\frac{1}{2}$. The theorem is proved.

5.7. Corollary. *If $\omega(\pi) > 2\mu + b - 1$, then*

$$(1) \quad \int_{\Omega \oplus V_1 \oplus V_1} ||K_\pi(\xi, u; v_0)||^2 \Delta(\xi)^{-2\mu-b} e^{-(u|u)} e^{-(v_0|v_0)} d\xi du dv_0 < \infty,$$

or

$$(2) \quad \int_{(AK_{-\frac{1}{2}}) \times V_1} ||K_\pi(a k(u); v_0)||^2 \Delta(a e)^{-\mu-b} e^{-(u|u)} e^{-(v_0|v_0)} da du dv_0 < \infty.$$

5.8. Remark. For the rest of the paper, we assume that π is an ifdh representation of $K^{\mathbb{C}}$ satisfying

$$(1) \quad \omega(\pi) > 2\mu + b - 1.$$

6. UNITARILY EQUIVALENT HILBERT SPACES

In this section, we construct Hilbert spaces $\mathfrak{S}\mathfrak{F}_{\Gamma}$ and \mathcal{H}_{π} of vector-valued functions. $\mathfrak{S}\mathfrak{F}_{\Gamma}$ is a space of π -covariant measurable functions on the group $AK_{-\frac{1}{2}}$, and \mathcal{H}_{π} is a space of holomorphic functions on the Siegel domain Σ of type II. We define an intertwining operator \mathcal{T} that establishes their unitary equivalence. We define the Hankel transform \mathcal{K}_{π} corresponding to the Bessel function K_{π} and we prove that the Hankel transform is a unitary operator of $\mathfrak{S}\mathfrak{F}_{\Gamma}$ to \mathcal{H}_{π} . We also study the reproducing kernel property of \mathcal{H}_{π} .

6.1. The space $\mathfrak{S}\mathfrak{F}_{\Gamma}$. By Theorem 4.4, the integral

$$(1) \quad \begin{aligned} \Gamma_{\pi} &= \Gamma_{\pi}(-\mu - b) \\ &= \int_{\Omega \oplus V_1} e^{-(\xi|e)} e^{-(u_0|u_0)} \pi(k(i u_0)^* r(\xi) k(i u_0))^{-1} \Delta(\xi)^{-2\mu-b} d\xi du_0 \end{aligned}$$

converges absolutely and defines a positive definite operator on \mathcal{V}_{π} . By (4.5.2)

$$\pi(l)\Gamma_{\pi} = \Gamma_{\pi}\pi(l)$$

for all $l \in L$.

Let $\mathfrak{S}\mathfrak{F}_{\pi}(AK_{-\frac{1}{2}})$ be the space of all measurable functions $f : AK_{-\frac{1}{2}} \rightarrow \mathcal{V}_{\pi}$ such that

$$(2) \quad f(a k(u_0) l) = \pi(l)^{-1} f(a k(u_0))$$

for $a \in A$, $u_0 \in V_1$ and $l \in L$, f is holomorphic in u_0 , and

$$(3) \quad \int_{AK_{-\frac{1}{2}}} \|f(a k(u_0))\|^2 e^{-(u_0|u_0)} da du_0 < \infty.$$

Then $\mathfrak{S}\mathfrak{F}_{\pi}(AK_{-\frac{1}{2}})$ is a Hilbert space with respect to the inner product

$$(4) \quad \langle f_1 | f_2 \rangle = \int_{AK_{-\frac{1}{2}}} (\Gamma_{\pi} f_1(a k(u_0)) | f_2(a k(u_0))) e^{-(u_0|u_0)} da du_0$$

and denote the resulting Hilbert space by $\mathfrak{S}\mathfrak{F}_{\Gamma}$. $\mathfrak{S}\mathfrak{F}_{\Gamma}$ is a restricted Schrödinger-Fock space in the sense of [Di1].

By (3.10.1) and (3.10.2), the space $\mathfrak{S}\mathfrak{F}_{\Gamma}$ can be identified with the space of all measurable functions $f : \Omega \oplus V_1 \rightarrow \mathcal{V}_{\pi}$ such that f is holomorphic in V_1 , and

$$(5) \quad \int_{\Omega \oplus V_1} \|f(\xi, u_0)\|^2 e^{-(u_0|u_0)} \Delta(\xi)^{-\mu} d\xi du_0 < \infty.$$

By the Laplace transform introduced in Section 4.6, $\mathfrak{S}\mathfrak{F}_{\Gamma}$ is unitarily equivalent to a space of holomorphic functions on Σ .

6.2. Theorem. For any $f \in \mathfrak{S}\mathfrak{F}_{\Gamma}$,

$$(1) \quad F(z, u) = \pi^{-\frac{n}{2}} \int_{AK_{-\frac{1}{2}}} e^{i(z|ae)} e^{(u|au_0)} e^{-(u_0|u_0)} \pi(a k(i u_0))^{*-1} f(a k(u_0)) da du_0$$

is absolutely convergent for all $(z, u) \in \Sigma$ and defines a holomorphic function $F : \Sigma \rightarrow \mathcal{V}_\pi$. Moreover,

$$(2) \quad \begin{aligned} & \int_{\Sigma} \|\pi(r(2y - B(u, \bar{u}))^{\frac{1}{2}} k(-i u))^*{}^{-1} F(x + iy, u)\|^2 \Delta(2y - B(u, \bar{u}))^{-2\mu-b} dx dy du \\ &= \int_{AK_{-\frac{1}{2}}} (\Gamma_\pi f(a k(u_0)) | f(a k(u_0))) e^{-(u_0|u_0)} da du_0. \end{aligned}$$

Proof. As in Theorem 4.4, we write $z = x + i(t + \frac{1}{2}B(u, \bar{u}))$. Then

$$(3) \quad \begin{aligned} & \int_{AK_{-\frac{1}{2}}} |e^{i(z|ae)} e^{(u|au_0)} e^{-(u_0|u_0)}| \|\pi(a k(i u_0))^*{}^{-1} f(a k(u_0))\| da du_0 \\ & \leq \int_{AK_{-\frac{1}{2}}} e^{-(t|ae)} e^{-\frac{1}{2}(u_0-a u|u_0-a u)} e^{-\frac{1}{2}(u_0|u_0)} \\ & \quad \times \|\pi(a k(i u_0))^*{}^{-1}\| \|f(a k(u_0))\| da du_0 \\ & \leq \left(\int_{AK_{-\frac{1}{2}}} e^{-2(t|ae)} e^{-(u_0-a u|u_0-a u)} \|\pi(a k(i u_0))^*{}^{-1}\|^2 da du_0 \right)^{\frac{1}{2}} \\ & \quad \times \left(\int_{AK_{-\frac{1}{2}}} \|f(a k(u_0))\|^2 e^{-(u_0|u_0)} da du_0 \right)^{\frac{1}{2}}. \end{aligned}$$

The first integral on the right side is finite by Theorem 4.4. The second integral is finite by (6.1.3). Thus the integral (1) is absolutely convergent and F is well defined. Applying Corollary 4.9, which is also valid for \mathcal{V}_π -valued functions, with $\alpha = -\mu - b$ and

$$\phi(a k(u_0)) = \Delta(a e)^{-\mu} f(a k(u_0)),$$

we obtain that F is holomorphic in Σ and (2) is valid.

6.3. The space \mathcal{H}_π . Denote by \mathcal{H}_π the space of all functions F of the integral form (6.2.1) for $f \in \mathfrak{S}\mathfrak{F}_\Gamma$, where $(z, u) \in \Sigma$. By Theorem 6.2, \mathcal{H}_π is a Hilbert space with the norm given by

$$(1) \quad \begin{aligned} \|F\|^2 &= \int_{\Sigma} \|\pi(r(2y - B(u, \bar{u}))^{\frac{1}{2}} k(-i u))^*{}^{-1} F(x + iy, u)\|^2 \\ & \quad \times \Delta(2y - B(u, \bar{u}))^{-2\mu-b} dx dy du. \end{aligned}$$

By (3.10.1) and (3.10.2), $f \in \mathfrak{S}\mathfrak{F}_\Gamma$ is identified with a function, still denoted by f , on $\Omega \oplus V_1$, and

$$(2) \quad \begin{aligned} F(z, u) &= \pi^{-\frac{n}{2}} \int_{\Omega \oplus V_1} e^{i(z|\xi)} e^{(\xi^{\frac{1}{2}} \circ u|u_0)} e^{-(u_0|u_0)} \pi(r(\xi^{\frac{1}{2}})k(i u_0))^*{}^{-1} \\ & \quad \times f(\xi, u_0) \Delta(\xi)^{-\mu} d\xi du_0. \end{aligned}$$

6.4. Theorem. The mapping $\mathcal{T} : f \mapsto F$ given by (6.2.1) is a unitary map of $\mathfrak{S}\mathfrak{F}_\Gamma$ onto \mathcal{H}_π . Furthermore, \mathcal{H}_π is the space of all \mathcal{V}_π -valued holomorphic functions F

on Σ satisfying

$$(1) \quad \int_{\Sigma} \|\pi(r(2y - B(u, \bar{u}))^{\frac{1}{2}} k(-i u))^*{}^{-1} F(x + iy, u)\|^2 \\ \times \Delta(2y - B(u, \bar{u}))^{-2\mu-b} dx dy du < \infty.$$

Proof. That \mathcal{T} is a unitary map follows from Theorem 6.2. Conversely, let F be a \mathcal{V}_{π} -valued holomorphic function on Σ satisfying (1). By [RV1], Theorem 5.11, the integral

$$\hat{F}(\xi, u) = \int_J F(x + iy, u) e^{-i(x+iy|\xi)} dx$$

is defined for almost all $(\xi, u) \in \Omega \oplus V_1$ and is independent of y . By [RV1], Lemmas 5.13 and 5.14, $\hat{F}(\xi, u)$ is equal to zero for almost all $\xi \notin \Omega$, $u \in V_1$, and

$$\int_{V_1} e^{-(\xi^{\frac{1}{2}} \circ u | \xi^{\frac{1}{2}} \circ u)} \|\hat{F}(\xi, u)\|^2 du < \infty$$

for almost all $\xi \in \Omega$. Thus, the integrals

$$\int_{V_1} e^{(u_0 | \xi^{\frac{1}{2}} \circ u)} e^{-(\xi^{\frac{1}{2}} \circ u | \xi^{\frac{1}{2}} \circ u)} \pi(r(\xi^{\frac{1}{2}}) k(i \xi^{\frac{1}{2}} \circ u)) \hat{F}(\xi, u) du$$

and

$$f(\xi, u_0) = 2^{-n_1} \pi^{-(\frac{n}{2}+rb)} \int_{J \oplus V_1} e^{-i(x+i(t+\frac{1}{2}B(u, \bar{u}))|\xi)} e^{(u_0 | \xi^{\frac{1}{2}} \circ u)} e^{-(\xi^{\frac{1}{2}} \circ u | \xi^{\frac{1}{2}} \circ u)} \\ (2) \quad \times \Delta(\xi)^b \pi(r(\xi^{\frac{1}{2}}) k(i \xi^{\frac{1}{2}} \circ u)) F(x + i(t + \frac{1}{2}B(u, \bar{u})), u) dx du$$

exist for almost all $\xi \in \Omega$, and the integral (2) is independent of the choice of $t > 0$. Similar to Proposition 4.7, one can see that $\mathcal{T}f$ given by (6.3.2) is F . Thus, \mathcal{T} is an onto mapping.

We now define the Hankel transform.

6.5. The Hankel transform. Let K_{π} be the Bessel function defined by (5.4.1). Let $(\mathfrak{S}\mathfrak{F}_{\Gamma})^0$ be the space of functions $f \in \mathfrak{S}\mathfrak{F}_{\Gamma}$ such that

$$(1) \quad \int_{AK_{-\frac{1}{2}}} \|f(a k(u_0))\|^2 e^{-(u_0|u_0)} \Delta(a e)^{\mu+b} da du_0 < \infty.$$

(1) is equivalent to the condition

$$(2) \quad \int_{\Omega \oplus V_1} \|f(\xi, u_0)\|^2 e^{-(u_0|u_0)} \Delta(\xi)^b d\xi du_0 < \infty.$$

Clearly, $(\mathfrak{S}\mathfrak{F}_{\Gamma})^0$ is dense in $\mathfrak{S}\mathfrak{F}_{\Gamma}$. For $f \in (\mathfrak{S}\mathfrak{F}_{\Gamma})^0$,

$$\int_{AK_{-\frac{1}{2}}} \|K_{\pi}(a_0^* a k(u_0); v_0) f(a_0 k(v_0))\|^2 e^{-(v_0|v_0)} da_0 dv_0 \\ \leq \left(\int_{AK_{-\frac{1}{2}}} \|K_{\pi}(a_0^* a k(u_0); v_0)\|^2 \Delta(a_0 e)^{-\mu-b} e^{-(v_0|v_0)} da_0 dv_0 \right)^{\frac{1}{2}} \\ \times \left(\int_{AK_{-\frac{1}{2}}} \|f(a_0 k(v_0))\|^2 \Delta(a_0 e)^{\mu+b} e^{-(v_0|v_0)} da_0 dv_0 \right)^{\frac{1}{2}}.$$

By (5.7.2)

$$\int_{AK_{-\frac{1}{2}}} \|K_{\pi}(a_0^* k(u_0); v_0)\|^2 \Delta(a_0 e)^{-\mu-b} e^{-(v_0|v_0)} da_0 dv_0 < \infty$$

for almost all $u_0 \in V_1$. By a variable change,

$$\begin{aligned} & \left(\int_{AK_{-\frac{1}{2}}} \|K_{\pi}(a_0^* a k(u_0); v_0)\|^2 \Delta(a_0 e)^{-\mu-b} e^{-(v_0|v_0)} da_0 dv_0 \right)^{\frac{1}{2}} \\ & \leq \left(\int_{AK_{-\frac{1}{2}}} \|K_{\pi}(a_0^* k(u_0); v_0)\|^2 \Delta(a_0 e)^{-\mu-b} e^{-(v_0|v_0)} da_0 dv_0 \right)^{\frac{1}{2}} \Delta(a e)^{\frac{1}{2}(\mu+b)}. \end{aligned}$$

Thus, the following integral transform is well defined.

6.6. Definition. For any $f \in (\mathfrak{S}_{\Gamma})^0$, the *Hankel transform* \mathcal{K}_{π} corresponding to the Bessel function K_{π} is defined by

$$(1) \quad (\mathcal{K}_{\pi} f)(a k(u_0)) = \int_{AK_{-\frac{1}{2}}} K_{\pi}(a_0^* a k(u_0); v_0) f(a_0 k(v_0)) e^{-(v_0|v_0)} da_0 dv_0.$$

6.7. Theorem. For $F \in \mathcal{H}_{\pi}$ denote

$$(2) \quad F^{\#}(z, u) = \pi(z) \pi(k(u)) F(-z^{-1}, z^{-1} \circ u).$$

Then

- (a) The mapping $\mathcal{U} : F \mapsto F^{\#}$ is a unitary operator on \mathcal{H}_{π} .
- (b) The operator \mathcal{K}_{π} on $(\mathfrak{S}_{\Gamma})^0$ extends uniquely to a unitary operator of \mathfrak{S}_{Γ} .
- (c) The unitary operator \mathcal{T} given by (6.2.1) intertwines \mathcal{K}_{π} and \mathcal{U} ; i.e.,

$$(2) \quad \mathcal{K}_{\pi} = \mathcal{T}^{-1} \mathcal{U} \mathcal{T}.$$

Proof. If $F \in \mathcal{H}_{\pi}$, then

$$\begin{aligned} & \int_{\Sigma} \|\pi(r(2y - B(u, \bar{u}))^{\frac{1}{2}} k(-i u))^* F^{\#}(x + iy, u)\|^2 \\ & \quad \times \Delta(2y - B(u, \bar{u}))^{-2\mu-b} dx dy du \\ & = \int_{\Sigma} \|\pi(r(2y - B(u, \bar{u}))^{\frac{1}{2}} k(-i u))^* \pi(z) \pi(k(u)) F(-z^{-1}, z^{-1} \circ u)\|^2 \\ (3) \quad & \quad \times \Delta(2y - B(u, \bar{u}))^{-2\mu-b} dx dy du \\ & = \int_{\Sigma} \text{tr} \left\{ F(-z^{-1}, z^{-1} \circ u)^* \pi(k(u))^* \pi(z)^* \pi(k(i u)) \pi(2y - B(u, \bar{u}))^{-1} \right. \\ & \quad \left. \times \pi(k(i u))^* \pi(z) \pi(k(u)) F(-z^{-1}, z^{-1} \circ u) \right\} \Delta(2y - B(u, \bar{u}))^{-2\mu-b} dx dy du. \end{aligned}$$

Changing variables $z \mapsto -z^{-1}$, $u \mapsto -z^{-1} \circ u$ and applying Lemma 5.5, (3) is equal to

$$\begin{aligned} & \int_{\Sigma} \text{tr} \left\{ F(z, u)^* \pi(k(i u)) r(2y - B(u, \bar{u}))^{-1} k(i u)^* F(z, u) \right\} \\ & \quad \times \Delta(2y - B(u, \bar{u}))^{-2\mu-b} dx dy du \\ & = \|F\|^2. \end{aligned}$$

Thus, $F^{\#} \in \mathcal{H}_{\pi}$, and \mathcal{U} is a unitary operator. (a) is proved.

Let $f \in (\mathfrak{S}\mathfrak{F}_\Gamma)^0$ and set

$$(4) \quad f^\#(a k(u_0)) = (\mathcal{K}_\pi f)(a k(u_0)).$$

It is clear that $f^\#$ satisfies the π -covariant condition (6.1.2).

We prove next that $f^\# \in \mathfrak{S}\mathfrak{F}_\Gamma$.

Formally constructing $\mathcal{T}f^\#$, we have

$$\begin{aligned} & (\mathcal{T}f^\#)(z, u) \\ &= \pi^{-\frac{n}{2}} \int_{AK_{-\frac{1}{2}}} e^{i(z|ae)} e^{(u|au_0)} e^{-(u_0|u_0)} \pi(a k(i u_0))^{*-1} f^\#(a k(u_0)) da du_0 \\ &= \pi^{-\frac{n}{2}} \int_{AK_{-\frac{1}{2}}} e^{i(z|ae)} e^{(u|au_0)} e^{-(u_0|u_0)} \pi(a k(i u_0))^{*-1} \\ &\quad \times \left(\int_{AK_{-\frac{1}{2}}} K_\pi(a_0^* a k(u_0); v_0) f(a_0 k(v_0)) e^{-(v_0|v_0)} da_0 dv_0 \right) da du_0 \\ &= \pi^{-\frac{n}{2}} \int_{AK_{-\frac{1}{2}}} \left(\int_{AK_{-\frac{1}{2}}} e^{i(z|ae)} e^{(u|au_0)} e^{-(u_0|u_0)} \pi(a k(i u_0))^{*-1} \right. \\ (5) \quad &\quad \left. \times K_\pi(a_0^* a k(u_0); v_0) da du_0 \right) f(a_0 k(v_0)) e^{-(v_0|v_0)} da_0 dv_0. \end{aligned}$$

By (5.4.2), (4.8.2) and variable changes, we have

$$\begin{aligned} & \int_{AK_{-\frac{1}{2}}} e^{i(z|ae)} e^{(u|au_0)} e^{-(u_0|u_0)} \pi(a k(i u_0))^{*-1} K_\pi(a_0^* a k(u_0); v_0) da du_0 \\ &= e^{-i(z^{-1}|a_0 e)} e^{(z^{-1}|B(a_0 v_0, \bar{u}))} \pi(z) \pi(a_0^*)^{-1} \pi(k(a_0^{-1} u)) \pi(k(-i v_0))^* \\ &= e^{-i(z^{-1}|a_0 e)} e^{(z^{-1} \circ u | a_0 v_0)} \pi(z) \pi(k(u)) \pi(a_0^*)^{-1} \pi(k(-i v_0))^*. \end{aligned}$$

Thus,

$$\begin{aligned} & (\mathcal{T}f^\#)(z, u) = \pi^{-\frac{n}{2}} \int_{AK_{-\frac{1}{2}}} e^{-i(z^{-1}|a_0 e)} e^{(z^{-1} \circ u | a_0 v_0)} \\ &\quad \times \pi(z) \pi(k(u)) \pi(a_0 k(i v_0))^{*-1} f(a_0 k(v_0)) e^{-(v_0|v_0)} da_0 dv_0 \\ &= \pi(z) \pi(k(u)) F(-z^{-1}, z^{-1} \circ u); \end{aligned}$$

i.e., formally

$$(6) \quad F^\# = \mathcal{T}f^\#.$$

Applying Corollary 4.9 with

$$\phi(a k(u_0)) = f^\#(a k(u_0)) \Delta(a e)^{-\mu},$$

we have

$$\begin{aligned}
 & \int_{AK_{-\frac{1}{2}}} (\Gamma_{\pi} f^{\#}(a k(u_0)) | f^{\#}(a k(u_0))) e^{-(u_0|u_0)} da du_0 \\
 &= \int_{\Sigma} \| \pi(r(2y - B(u, \bar{u}))^{\frac{1}{2}} k(-i u))^* F^{\#}(x + iy, u) \|^2 \\
 (7) \quad & \quad \times \Delta(2y - B(u, \bar{u}))^{-2\mu-b} dx dy du \\
 &= \int_{\Sigma} \| \pi(r(2y - B(u, \bar{u}))^{\frac{1}{2}} k(-i u))^* F(x + iy, u) \|^2 \\
 & \quad \times \Delta(2y - B(u, \bar{u}))^{-2\mu-b} dx dy du \\
 &< \infty
 \end{aligned}$$

by (a). Thus, $f^{\#} \in \mathfrak{S}\mathfrak{F}_{\Gamma}$.

By (4) and (6),

$$\mathcal{K}_{\pi} f = \mathcal{T}^{-1} \mathcal{U} \mathcal{T} f$$

for all $f \in (\mathfrak{S}\mathfrak{F}_{\Gamma})^0$. Hence \mathcal{K}_{π} agrees with the unitary operator $\mathcal{T}^{-1} \mathcal{U} \mathcal{T}$ on a dense subspace and \mathcal{K}_{π} extends uniquely from $(\mathfrak{S}\mathfrak{F}_{\Gamma})^0$ to a unitary operator on $\mathfrak{S}\mathfrak{F}_{\Gamma}$. The theorem is proved.

We now study the reproducing kernel property of the space \mathcal{H}_{π} .

6.8. Reproducing kernels. By (6.2.3), $F = \mathcal{T} f$ satisfies that

$$\begin{aligned}
 |F(z, u)| &\leq \|f\| \left(\int_{AK_{-\frac{1}{2}}} e^{-2(t|ae)} e^{-(u_0 - a^* u | u_0 - a^* u)} \| \pi(a k(i u_0))^* \|^2 da du_0 \right)^{\frac{1}{2}} \\
 &\leq \|F\| \left(\int_{AK_{-\frac{1}{2}}} e^{-2(t|ae)} e^{-(u_0 - a^* u | u_0 - a^* u)} \| \pi(a k(i u_0))^* \|^2 da du_0 \right)^{\frac{1}{2}}
 \end{aligned}$$

for $f \in \mathfrak{S}\mathfrak{F}_{\Gamma}$. Thus, for each $(z, u) \in \Sigma$ the point evaluation

$$(1) \quad E_{(z,u)} : F \mapsto F(z, u)$$

is a continuous linear operator from \mathcal{H}_{π} to \mathcal{V}_{π} . Then by [Ku], \mathcal{H}_{π} is characterized by its reproducing kernel

$$(2) \quad Q_{\pi}((z, u), (z', u')) = E_{(z,u)} E_{(z',u')}^*$$

in the sense that

$$(3) \quad (F(z', u') | v) = \langle F | Q_{\pi}(\cdot, (z', u')) v \rangle$$

for all $F \in \mathcal{H}_{\pi}$, $v \in \mathcal{V}_{\pi}$ and $(z', u') \in \Sigma$.

We now obtain an integral representation for the reproducing kernel of the space \mathcal{H}_{π} .

6.9. Theorem. For each $(z', u') \in \Sigma$, the integral

$$\begin{aligned}
 & F_{(z',u')}(z, u) \\
 &= \pi^{-n} \int_{AK_{-\frac{1}{2}}} e^{i(z-z'|ae)} e^{(u|a u_0)} e^{(a u_0|u')} e^{-(u_0|u_0)} \\
 (1) \quad & \times \pi(a k(i u_0))^* \Gamma_{\pi}^{-1} \pi(a k(i u_0))^{-1} da du_0
 \end{aligned}$$

converges absolutely and defines a holomorphic function $F_{(z', u')}$ of (z, u) . $F_{(z', u')}$ is the reproducing kernel of \mathcal{H}_π in the sense of (6.8.3); i.e.,

$$(2) \quad (F(z', u')|v) = \langle F|F_{(z', u')}(\cdot) v \rangle,$$

for all $F \in \mathcal{H}_\pi$ and $v \in \mathcal{V}_\pi$.

Proof. Let $f \in \mathfrak{S}\mathfrak{F}_\Gamma$ such that $F = \mathcal{T}f$. Then by (6.2.1) and (6.1.4), we have

$$\begin{aligned} & (F(z', u')|v) \\ &= \left(\pi^{-\frac{n}{2}} \int_{AK_{-\frac{1}{2}}} e^{i(z'|ae)} e^{(u'|au_0)} e^{-(u_0|u_0)} \pi(ak(iu_0))^{*-1} f(ak(u_0)) da du_0 \Big| v \right) \\ &= \langle f|f_{(z', u')}(\cdot) v \rangle \end{aligned}$$

for $(z', u') \in \Sigma$ and $v \in \mathcal{V}_\pi$, where

$$f_{(z', u')}(ak(u_0)) = \pi^{-\frac{n}{2}} e^{-i(ae|z')} e^{(au_0|u')} \Gamma_\pi^{-1} \pi(ak(iu_0))^{-1}.$$

It is clear that the function $f_{(z', u')}$ satisfies (6.1.2). By Theorem 4.4, it satisfies (6.1.3). Thus, $f_{(z', u')}(\cdot)v \in \mathfrak{S}\mathfrak{F}_\Gamma$ for $v \in \mathcal{V}_\pi$.

Define

$$F_{(z', u')} v = \mathcal{T}f_{(z', u')} v.$$

for $v \in \mathcal{V}_\pi$. Then, $F_{(z', u')}$ can be expressed by (1) and

$$(F(z', u')|v) = \langle f|f_{(z', u')}(\cdot) v \rangle = \langle F|F_{(z', u')}(\cdot) v \rangle$$

for $v \in \mathcal{V}_\pi$. The theorem is proved.

7. HOLOMORPHIC DISCRETE SERIES

In this section, we construct equivalent irreducible unitary representations S_π and T_π of G on the spaces $\mathfrak{S}\mathfrak{F}_\Gamma$ and \mathcal{H}_π , respectively, which are intertwined by the operator \mathcal{T} . We show that S_π and T_π are irreducible and square-integrable, and that they are in the holomorphic discrete series of G . We compute the reproducing kernel of \mathcal{H}_π explicitly.

7.1. Definition. We define the \mathcal{E} -valued function m_π by

$$(1) \quad m_\pi(g, (z, u)) = \pi(m(g, (z, u)))$$

for $(g, (z, u)) \in G^\mathbb{C} \times (V_2 \oplus V_1)$, where m is given by (3.7.2). Then m_π satisfies the multiplier identity (3.7.3).

For $F \in \mathcal{H}_\pi$ and $g \in G$, define

$$(2) \quad (T_\pi(g)F)(z, u) = m_\pi(g, (z, u)) F(g^{-1} \cdot (z, u))$$

for $(z, u) \in \Sigma$.

7.2. Theorem. T_π is a unitary representation of G on \mathcal{H}_π . For generators a , $R(u_0)$, $R(x_0)$, j of G given by (3.6.4), (3.6.6), (3.6.7) and (3.6.8), T_π acts on \mathcal{H}_π as follows:

$$(1) \quad \begin{cases} (T_\pi(a)F)(z, u) = \pi(a)F(a^{-1}z, a^{-1}u), \\ (T_\pi(R(u_0)F)(z, u) = \pi(k(-iu_0))^* F(z - iB(u, \bar{u}_0) + \frac{i}{2}B(u_0, \bar{u}_0), u - u_0), \\ (T_\pi(R(x_0)F)(z, u) = F(z - x_0, u), \\ (T_\pi(j)F)(z, u) = \pi(z)\pi(k(u))F(-z^{-1}, z^{-1} \circ u). \end{cases}$$

Proof. Since m_π satisfies the multiplier identity (3.7.3), T_π is a representation. That is,

$$T_\pi(g_1 g_2) F = T_\pi(g_1) T_\pi(g_2) F$$

for all $g_1, g_2 \in G$ and $F \in \mathcal{H}_\pi$. Formula (1) follows from (7.1.2) directly. By the norm (6.3.1), $T_\pi(a)$, $T_\pi(R(u_0))$, $T_\pi(R(x_0))$ are unitary operators on \mathcal{H}_π . That $T_\pi(j)$ is unitary follows from Theorem 6.7.

7.3. Theorem. *The following formulas define a unitary representation of G on \mathfrak{S}_Γ .*

$$(1) \quad \begin{cases} (S_\pi(a_0) f)(a k(u)) = f(a_0^* a k(u)), \\ (S_\pi(R(u_0)) f)(a k(u)) = e^{-\frac{1}{2}(a^* u_0 | a^* u_0)} e^{(u | a^* u_0)} f(a k(u - a^* u_0)), \\ (S_\pi(R(x_0)) f)(a k(u)) = e^{-i(x_0 | a e)} f(a k(u)), \\ (S_\pi(j) f)(a k(u)) = (\mathcal{K}_\pi f)(a k(u)). \end{cases}$$

Moreover, $S_\pi = \mathcal{T}^{-1} T_\pi \mathcal{T}$.

Proof. $S_\pi = \mathcal{T}^{-1} T_\pi \mathcal{T}$ follows directly from (1), (7.2.1), (6.2.1), (6.7.2) and a computation. Since T_π is a unitary representation of G , so is S_π .

By [Di4, Theorem], we now show that the representation T_π of G is irreducible.

7.4. Theorem. *The representation T_π of G on \mathcal{H}_π is irreducible.*

Proof. By Section 6.8, the point evaluation $E_{(z,u)}$ is a continuous operator from \mathcal{H}_π to \mathcal{V}_π for each $(z, u) \in \Sigma$.

Let K be the maximal compact subgroup of G' and c be the Cayley transform given by (2.7.3). Note that K is the isotropy subgroup of 0 in G' and that $c \cdot 0 = i e$. By (3.7.3), for $k \in K$ we have

$$\begin{aligned} m(c k c^{-1}, i e) &= m(c, i e) m(k c^{-1}, c^{-1} \cdot i e) \\ &= m(c, i e) m(k c^{-1}, 0) \\ &= m(c, i e) m(k, 0) m(c^{-1}, k^{-1} \cdot 0) \\ (1) \quad &= m(c, i e) k m(c^{-1}, 0). \end{aligned}$$

Taking k in (1) the identity element I of K , we have

$$m(c, i e) m(c^{-1}, 0) = m(I, i e) = I.$$

Thus, $m(c, i e)^{-1} = m(c^{-1}, 0)$. Suppose that $l \in \mathcal{E}$ satisfies

$$(2) \quad m_\pi(c k c^{-1}, i e) l m_\pi(c k c^{-1}, i e)^* = l$$

for all $k \in K$. By (1) and (7.1.1), we have

$$m_\pi(c, i e) \pi(k) m_\pi(c^{-1}, 0) l m_\pi(c^{-1}, 0)^* \pi(k^*) m_\pi(c, i e)^* = l,$$

or

$$\pi(k) m_\pi(c^{-1}, 0) l m_\pi(c^{-1}, 0)^* \pi(k^*) = m_\pi(c^{-1}, 0) l m_\pi(c^{-1}, 0)^*.$$

Since π is an irreducible representation of K , $m_\pi(c^{-1}, 0) l m_\pi(c^{-1}, 0)^*$ is a constant. By the Theorem of [Di4], T_π is irreducible.

7.5. Corollary. *The representation S_π of G is irreducible.*

We now compute the reproducing kernel Q_π given by (6.8.2).

7.6. Theorem. *The reproducing kernel for \mathcal{H}_π is given explicitly by the formula*

$$(1) \quad Q_\pi((z, u), (z', u')) = \gamma_\pi \pi(k(iu)^* r(z - z'^*) k(iu)),$$

where γ_π is a positive constant

$$(2) \quad \gamma_\pi = \pi^{-n} d^{-1} \operatorname{tr}(\Gamma_\pi(0) \Gamma_\pi^{-1}),$$

d is the dimension of \mathcal{V}_π , and $\Gamma_\pi(0)$ and Γ_π are given by (4.3.2) and (6.1.1), respectively.

Proof. Let $E_{(z,u)}$ be the point evaluation on \mathcal{H}_π , which is a continuous linear operator of \mathcal{H}_π to \mathcal{V}_π by Section 6.8. Rewrite (7.1.2) as

$$(3) \quad E_{(z,u)} T_\pi(g) = m_\pi(g, (z, u)) E_{g^{-1} \cdot (z, u)}$$

for $g \in G$, $(z, u) \in \Sigma$. Since T_π is unitary,

$$T_\pi(g)^{-1} E_{(z,u)}^* = E_{g^{-1} \cdot (z, u)}^* m_\pi(g, (z, u))^*.$$

Thus,

$$(4) \quad E_{(z',u')} E_{(z,u)}^* = m_\pi(g, (z', u')) E_{g^{-1} \cdot (z', u')} E_{g^{-1} \cdot (z, u)}^* m_\pi(g, (z, u))^*$$

for $g \in G$, $(z, u), (z', u') \in \Sigma$.

Letting $(z, u) = (z', u')$ and $g^{-1} \cdot (z, u) = ie$ in (4), we have

$$(5) \quad E_{(z,u)} E_{(z,u)}^* = m_\pi(g, (z, u)) E_{ie} E_{ie}^* m_\pi(g, (z, u))^*.$$

Since m_π satisfies the multiplier identity (3.7.3),

$$\begin{aligned} I &= m_\pi(gg^{-1}, (z, u)) = m_\pi(g, (z, u)) m_\pi(g^{-1}, g^{-1} \cdot (z, u)) \\ &= m_\pi(g, (z, u)) m_\pi(g^{-1}, ie) \end{aligned}$$

and

$$(6) \quad m_\pi(g, (z, u)) = m_\pi(g^{-1}, ie)^{-1}.$$

In particular, taking $g = R(x) R(u) r(t^{\frac{1}{2}})$ with $(z, u) = (x + i(t + \frac{1}{2}B(u, \bar{u})), u)$, we have $g \cdot ie = (z, u)$. By (3.8.1)–(3.8.3), we have

$$\begin{aligned} m_\pi(g^{-1}, ie) &= m_\pi(r(t^{-\frac{1}{2}}) R(u)^{-1} R(-x), ie) \\ &= m_\pi(r(t^{-\frac{1}{2}}), ie) m_\pi(R(u)^{-1} R(-x), r(t^{\frac{1}{2}}) \cdot ie) \\ &= \pi(t^{-\frac{1}{2}}) m_\pi(R(u)^{-1}, it) m_\pi(R(-x), R(u) \cdot it) \\ (7) \quad &= \pi(t^{-\frac{1}{2}}) \pi(k(-iu))^*. \end{aligned}$$

By (5), (6) and (7),

$$(8) \quad E_{(z,u)} E_{(z,u)}^* = \pi(k(iu))^* \pi(t^{\frac{1}{2}}) E_{ie} E_{ie}^* \pi(t^{\frac{1}{2}}) \pi(k(iu)).$$

Letting $(z, u) = (z', u') = ie$ and $g = ckc^{-1}$ in (4), where $k \in K$ and c is the Cayley transform given by (2.7.3), we have

$$E_{ie} E_{ie}^* = m_\pi(ckc^{-1}, ie) E_{ie} E_{ie}^* m_\pi(ckc^{-1}, ie)^*.$$

Thus, $E_{ie} E_{ie}^*$ satisfies (7.4.2), and $m_\pi(c^{-1}, 0) E_{ie} E_{ie}^* m_\pi(c^{-1}, 0)^*$ is a constant γ_π . Substituting in (8), we have

$$E_{(z,u)} E_{(z,u)}^* = \gamma_\pi \pi(k(iu))^* \pi(t^{\frac{1}{2}}) m_\pi(c^{-1}, 0)^{-1} m_\pi(c^{-1}, 0)^{*^{-1}} \pi(t^{\frac{1}{2}}) \pi(k(iu)).$$

By (2.7.3), (3.7.1), (3.7.2) and (7.1.1), $m_\pi(c^{-1}, 0) = \pi(\frac{\sqrt{2}}{2}e)$ and $m_\pi(c^{-1}, 0)^{-1} = m_\pi(c^{-1}, 0)^{* -1} = \pi(\sqrt{2}e)$. Thus

$$\begin{aligned} E_{(z,u)} E_{(z,u)}^* &= \gamma_\pi \pi(k(iu))^* \pi(\sqrt{2}t^{\frac{1}{2}}) \pi(\sqrt{2}t^{\frac{1}{2}}) \pi(k(iu)) \\ &= \gamma_\pi \pi(k(iu))^* \pi(2t) \pi(k(iu)). \end{aligned}$$

By (6.8.2), (6.9.1), (4.3.2) and calculating $E_{ie} E_{ie}^*$, we have (2). Now (1) follows from [Ku, Lemma], and the theorem is proved.

We now study the square-integrability of S_π (and T_π).

7.7. Measure decomposition on G . Let dx and du be Lebesgue measures on J and V_1 , respectively, and da Haar measure on A . $R(u_0)$ and $R(x_0)$ commute, and generate a subgroup N of G . A normalizes N , and $M = NA$ is a subgroup of G .

$$(1) \quad \int_M f(m) dm = \int_A \int_{J \oplus V_1} f(R(x_0) R(u_0) a) dx_0 du_0 da$$

defines a right Haar measure on M , and this equality is valid for any nonnegative function f . Moreover, $d_l m = dm^{-1}$ is a left Haar measure on M and the Haar measure dg on G decomposes as $dg = d(m^{-1} j R(u_0) R(x_0)) = dm dx_0 du_0$. Thus the formula

$$(2) \quad \int_G f(g) dg = \int_M \int_{J \oplus V_1} f(m^{-1} j R(x_0) R(u_0)) dm dx_0 du_0$$

is valid for all nonnegative measurable functions f .

7.8. Theorem. For any f_1 and f_2 in $\mathfrak{S}\mathfrak{F}_\Gamma$,

$$\begin{aligned} & \int_M |\langle S_\pi(m) f_1 | f_2 \rangle|^2 dm \\ (1) \quad & \leq c \|f_1\|^2 \left(\int_{AK_{-\frac{1}{2}}} \|\Gamma_{\frac{1}{2}} f_2(a k(u_0))\|^2 e^{-(u_0|u_0)} \Delta(ae)^{-\mu-b} da du_0 \right) \end{aligned}$$

for some constant c .

Proof. By (6.1.4) and (7.3.1),

$$\begin{aligned} & \langle S_\pi(R(u_0) R(x_0) a_0) f_1 | f_2 \rangle \\ &= \int_{AK_{-\frac{1}{2}}} e^{-i(x_0|ae)} e^{-\frac{1}{2}(a^* u_0 | a^* u_0)} e^{(u|a^* u_0)} \\ & \times \operatorname{tr} \left\{ f_2(a k(u))^* \Gamma_\pi f_1(a_0^* a k(u - a^* u_0)) \right\} e^{-(u|u)} da du. \end{aligned}$$

By (3.10.1), (3.10.2), (7.7.1) and the Plancherel formula for functions on J ,

$$\begin{aligned}
& \int_M |\langle S_\pi(R(u_0) R(x_0) a) f_1 | f_2 \rangle|^2 dm \\
&= \int_A \int_{J \oplus V_1} \left| \int_{\Omega \oplus V_1} e^{-i(x_0|\xi)} e^{-\frac{1}{2}(\xi^{\frac{1}{2}} \circ u_0 | \xi^{\frac{1}{2}} \circ u_0)} e^{(u|\xi^{\frac{1}{2}} \circ u_0)} \right. \\
&\quad \times \operatorname{tr} \left\{ f_2(\xi, u)^* \Gamma_\pi f_1(a_0^* \xi^{\frac{1}{2}}, u - \xi^{\frac{1}{2}} \circ u_0) \right\} e^{-(u|u)} \Delta(\xi)^{-\mu} d\xi du \Big|^2 dx_0 du_0 da_0 \\
&= (2\pi)^{n_1} \int_{(\Omega \oplus V_1) \times A} \left| \int_{V_1} e^{-\frac{1}{2}(\xi \circ u_0 | u_0)} e^{(u|\xi^{\frac{1}{2}} \circ u_0)} \right. \\
&\quad \times \operatorname{tr} \left\{ f_2(\xi, u)^* \Gamma_\pi f_1(a_0^* \xi^{\frac{1}{2}}, u - \xi^{\frac{1}{2}} \circ u_0) \right\} e^{-(u|u)} du \Big|^2 \Delta(\xi)^{-2\mu} d\xi du_0 da_0 \\
&= (2\pi)^{n_1} \int_{A \times V_1 \times A} \left| \int_{V_1} e^{-\frac{1}{2}(a^* u_0 | a^* u_0)} e^{(u|a^* u_0)} \right. \\
&\quad \times \operatorname{tr} \left\{ f_2(a k(u))^* \Gamma_\pi f_1(a_0^* a k(u - a^* u_0)) \right\} e^{-(u|u)} du \Big|^2 \Delta(a e)^{-\mu} da_0 du_0 da.
\end{aligned}$$

By [Di2, Theorem 4.3] with the normalized measure $\pi^{-rb} du_0$ and variable changes $a^* u_0 \mapsto u_0$, $a_0^* a \mapsto a_0$,

$$\begin{aligned}
(2) \quad & \int_M |\langle S_\pi(R(u_0) R(x_0) a_0) f_1 | f_2 \rangle|^2 dm \\
&\leq (2\pi)^{n_1} \pi^{rb} \int_{A \times V_1 \times A} \operatorname{tr} \left\{ \left(\Gamma_\pi^{\frac{1}{2}} f_2(a k(u_0)) \right)^* \left(\int_{V_1} \|\Gamma_\pi^{\frac{1}{2}} f_1(a_0^* a k(u))\|^2 e^{-(u|u)} du \right) \right. \\
&\quad \left. \times \Gamma_\pi^{\frac{1}{2}} f_2(a k(u_0)) \right\} e^{-(u_0|u_0)} \Delta(a e)^{-\mu-b} da_0 du_0 da \\
&= (2\pi)^{n_1} \pi^{rb} \|f_1\|^2 \int_{AK_{-\frac{1}{2}}} \|\Gamma_\pi^{\frac{1}{2}} f_2(a_0 k(u_0))\|^2 e^{-(u_0|u_0)} \Delta(a_0 e)^{-\mu-b} da_0 du_0.
\end{aligned}$$

The theorem is proved.

7.9. Theorem. For any f_1 and f_2 in $\mathfrak{S}_{\mathfrak{F}_\Gamma}$,

$$\begin{aligned}
(1) \quad & \int_G |\langle S_\pi(g) f_1 | f_2 \rangle|^2 dg \\
&\leq c \|f_1\|^2 \|f_2\|^2 \int_{\Omega \oplus V_1 \oplus V_1} (\Gamma_\pi K_\pi(\xi, u; v_0) | K_\pi(\xi, u; v_0)) \\
&\quad \times \Delta(\xi)^{-2\mu-b} e^{-(u|u)} e^{-(v_0|v_0)} d\xi du dv_0
\end{aligned}$$

for some constant c .

Proof. By (7.7.2) and (7.8.2),

$$\begin{aligned}
 & \int_G |\langle S_\pi(g) f_1 | f_2 \rangle|^2 dg \\
 &= \int_{J \oplus V_1} \int_M |\langle S_\pi(m) f_2 | S_\pi(j R(u_0) R(x_0)) f_1 \rangle|^2 dm dx_0 du_0 \\
 &\leq 2^{n_1} \pi^n \|f_2\|^2 \int_{J \oplus V_1} \left(\int_{AK_{-\frac{1}{2}}} \left\| \Gamma_\pi^{\frac{1}{2}} S_\pi(j R(u_0) R(x_0)) f_1(a k(u)) \right\|^2 \right. \\
 &\quad \left. \times e^{-(u|u)} \Delta(a e)^{-\mu-b} da du \right) dx_0 du_0.
 \end{aligned}
 \tag{2}$$

We assume temporarily that f_1 lies in the dense subspace $(\mathfrak{S}\mathfrak{F}_\Gamma)^0$ of $\mathfrak{S}\mathfrak{F}_\Gamma$ for which \mathcal{K}_π is given by (6.6.1). Then

$$\begin{aligned}
 & (S_\pi(j R(u_0) R(x_0)) f_1)(a k(u)) \\
 &= \int_{AK_{-\frac{1}{2}}} K_\pi(a_0 a^* k(u); v_0) e^{-i(x_0|a_0 e)} e^{-\frac{1}{2}(a_0^* u_0 | a_0^* u_0)} \\
 &\quad \times e^{(v_0|a_0^* u_0)} f_1(a_0 k(v_0 - a_0^* u_0)) da_0 dv_0.
 \end{aligned}
 \tag{3}$$

By (2), (3), the Plancherel formula, and [Di2, Theorem 4.3],

$$\begin{aligned}
 & \int_G |\langle S_\pi(g) f_1 | f_2 \rangle|^2 dg \\
 &\leq c \|f_1\|^2 \|f_2\|^2 \int_{\Omega \oplus V_1 \oplus V_1} (\Gamma_\pi K_\pi(\xi, u; v_0) | K_\pi(\xi, u; v_0)) \\
 &\quad \times \Delta(\xi)^{-2\mu-b} e^{-(u|u)} e^{-(v_0|v_0)} d\xi du dv_0.
 \end{aligned}
 \tag{4}$$

This proves (1) on a dense subset $(\mathfrak{S}\mathfrak{F}_\Gamma)^0$ of $\mathfrak{S}\mathfrak{F}_\Gamma$, but it follows readily from Fatou's lemma that (1) holds for all $f_1, f_2 \in \mathfrak{S}\mathfrak{F}_\Gamma$, which completes the proof of the theorem.

7.10. Holomorphic discrete series. By Corollary 7.5 and Theorem 7.9, S_π (also T_π) are in the holomorphic discrete series of G . As shown in [RV1], the representations T_π constitute the holomorphic discrete series of representations of G . The condition $\omega(\pi) > 2\mu + b - 1$ (cf. (5.8.1)) is equivalent to Harish-Chandra's condition on the highest weight of π for the existence of the holomorphic discrete series.

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